Homework 8: Dynamics

EECS/ME/BioE C106A/206A Introduction to Robotics

Fall 2023

Note: This problem set includes a programming component. Your deliverables for this assignment are:

1. A PDF file submitted to the `HW8 (pdf)` Gradescope assignment with all your work and solutions to the written problems.
2. The `hw8.py` file submitted to the `HW8 (code)` Gradescope assignment with your implementation to the programming component.

Problem 1: Dynamics of a Mass-Spring System

Figure 1 shows a system involving a mass m and a spring with spring constant k on an incline. Pick a suitable set of generalized coordinates (you should only need one), and use Lagrangian dynamics to find the equations of motion of the mass-spring system. State the Inertia matrix, Coriolis matrix, and Gravity vector for this system (since this is a one dimensional problem, these will all just be scalars). What is the physical meaning of the generalized forces Υ in this case?

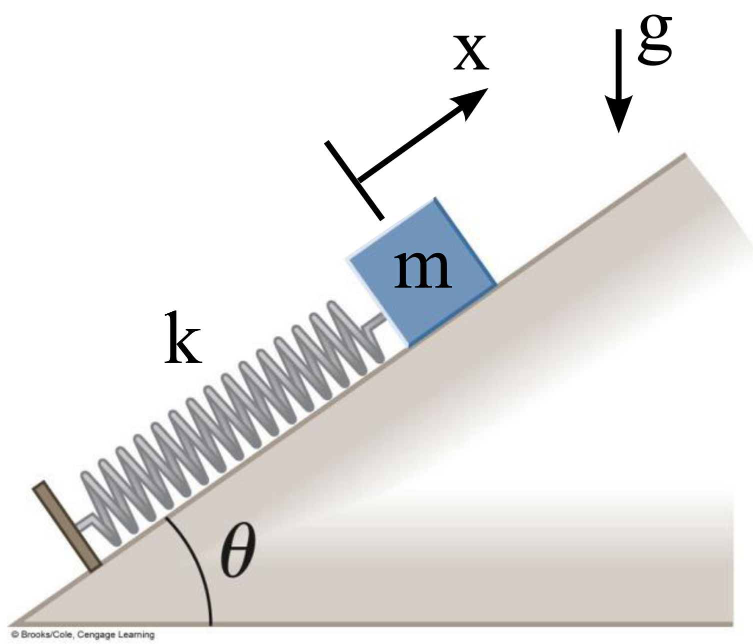
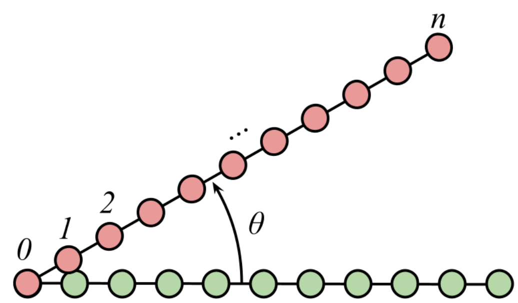


Figure 1: Mass-spring system on a slope.

Problem 2: Rotational Inertia

Let's shift our focus from the description of *particle* dynamics to *rigid body* dynamics. Rigid bodies are non-deformable masses composed of an *infinite* number of particles.

(a) Consider the following system, where we have particles $0, 1, 2, ..., n$, each of mass $m/(n+1)$. The i^{th} particle is at a distance il/n from the origin, and particle 0 is position at $(x, y) = (0, 0)$.



The entire system spins around the origin together with an angle θ. Calculate $r_i(\theta) = [x_i(\theta), y_i(\theta)]^T$, the position of the i^{th} particle when the system is at angle θ. Then, for $\frac{d\theta}{dt} = \dot{\theta}$, show that the time derivative of r_i is calculated:

$$\dot{r}_i = \frac{il}{n} \begin{bmatrix} -\dot{\theta}\sin\theta \\ \dot{\theta}\cos\theta \end{bmatrix} \tag{1}$$

(b) Prove the following formula by induction on n, where $n \geq 1$:

$$\sum_{i=0}^{n} i^2 = \frac{1}{6}n(n+1)(2n+1) \tag{2}$$

(c) Using part (a), we can calculate T_i, the kinetic energy of each particle in the system. By summing the T_is, we can find the total kinetic energy! Using (b), show that the total kinetic energy of the system is calculated:

$$\sum_{i=0}^{n} T_i = \frac{1}{12}ml^2\dot{\theta}^2\frac{2n+1}{n} \tag{3}$$

(d) Let's now take the limit of this kinetic energy as the number of particles approach infinity. Show that the following limits hold, where r_n is the position vector of particle n:

$$\sum_{i=0}^{\infty} T_i = \frac{1}{6}ml^2\dot{\theta}^2 \tag{4}$$

$$\sum_{i=0}^{\infty} m_i = m \tag{5}$$

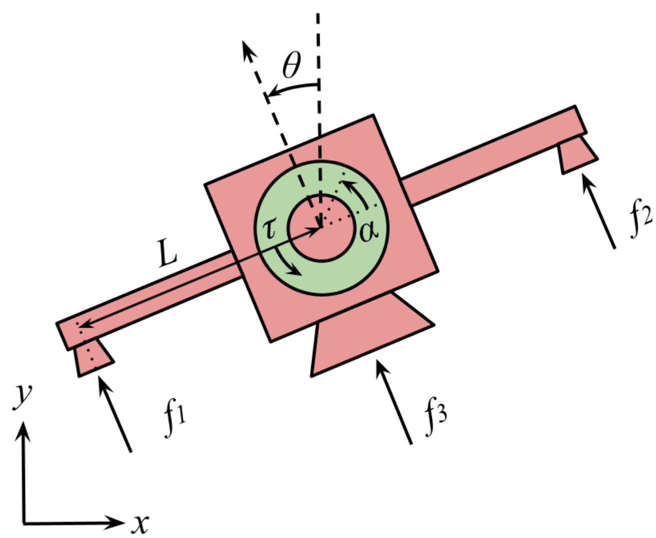
$$\lim_{n\to\infty} ||r_n|| = l \tag{6}$$

(e) In the limit, we've transformed our system of finite particles into a *rigid body*! The rotational kinetic energy of a rigid body is expressed $T = \frac{1}{2}I\dot{\theta}^2$, where I is the inertia. What is the inertia of the rigid bar system above in terms of m and l?

Problem 3: Spacecraft Dynamics

Your space startup, SpaceY, is back at it again! This time, you're planning an autonomous interplanetary mission from Earth to Mars, and are looking to model the dynamics of your spacecraft.

1. Your spacecraft, which is a small, autonomous probe, is pictured below:



   The center of mass of the body of the probe is given by (x, y) and its angle of orientation with respect to the vertical by an angle θ. Additionally, the probe has a *reaction wheel*, which may be spun up to control the orientation of the craft. This wheel has angle α with respect to the *body* of the probe. If the body has mass m_b and inertia I_b, and the reaction wheel has mass m_r and inertia I_r, find the total kinetic energy T of the system.

2. Your spacecraft has three thrusters with forces f_1, f_2, f_3 that apply a force along the direction of the body, and an electric motor which applies a torque τ to the flywheel. The thrusters f_1 and f_2 are located a distance L from the vehicle's mass center. Assuming no gravity, apply Lagrange's equations to find the equations of motion of the system.

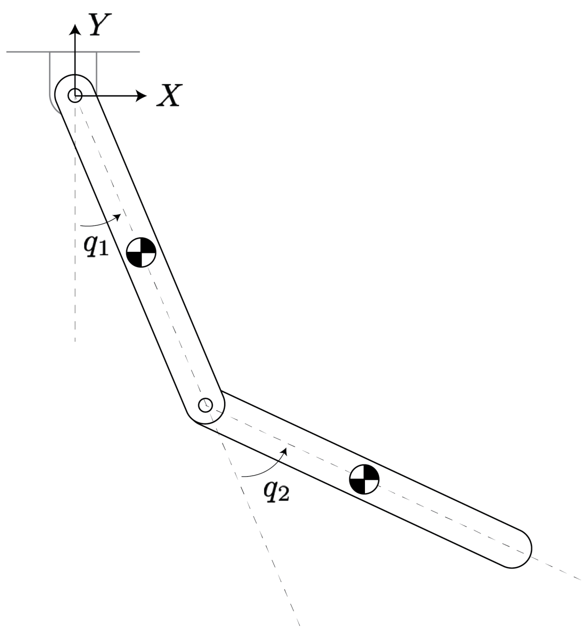
Now that we have equations of motion, we can construct a control law for this spacecraft! We know exactly what forces we need to apply to achieve a desired acceleration. We'll talk a lot more about control next week.

Problem 4: Robotic Legs & The Double Pendulum

Professor Sreenath wants to design a feedback controller for his robotic dog, the Unitree A1. Before he can do this, he needs to solve for a model for the dynamics of the robot's legs! The Unitree A1 has two leg joints - one at the hip and one at the knee.

Above: The Unitree A1 robotic dog

Because it has two joints, we can develop a simple model of the Unitree's legs using a system called the *double pendulum*.



As we can see, this system is composed of a double pendulum hanging from a rigidly fixed pivot. Both arms have length L, mass m, and moments of inertia I about the $z-$axis (pointing out of the plane of the paper) at the center of mass. Both arms have a uniform mass density so that their center of masses are at their centers. Gravity points in the negative y direction.

Computer algebra systems like SymPy, the MATLAB symbolic toolbox, and Mathematica lets you write code to perform symbolic mathematical computation. This is very useful when you need to perform some tedious algebra and want your final answer to be in terms of some variables. This includes taking integrals, derivatives, matrix exponentials, eigenvalues, etc.

To model the force of the motors on the different joints of the leg, we consider a torque τ applied at each of the joints. Let's apply the methods of Lagrangian dynamics to estimate the dynamics of the Unitree A1's leg! Pick a suitable set of generalized coordinates (you should need two). Fill out `hw8.py` to determine the Inertia matrix, Gravity vector, and a possible Coriolis matrix for this system. What is the physical meaning of the generalized forces Υ in this case?

The autograder will look at the variables defined in `hw8.py` to grade you on the correctness of

1. The kinetic energy T of the system.
2. The potential energy V of the system.
3. The necessary derivatives of the Lagrangian: $\frac{\partial L}{\partial q}$ and $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right)$.
4. The inertia matrix $M(q)$ of the system.
5. The Coriolis term $C(q, \dot{q})\dot{q}$ and your choice of Coriolis matrix $C(q, \dot{q})$.
6. The gravity vector $N(q, \dot{q})$.