Homework 7: Jacobians

EECS/ME/BioE C106A/206A Introduction to Robotics

Fall 2023

Problem 1: Jacobian for a 4DOF manipulator

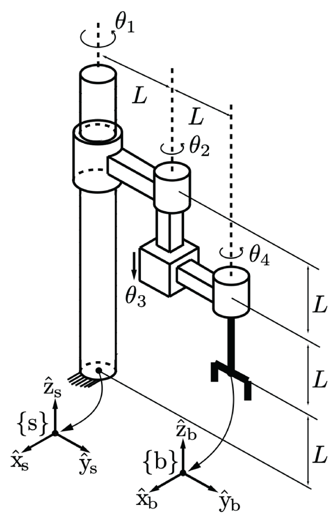

Figure 1: A four degree of freedom manipulator

Figure 1 shows a 4DOF manipulator with 3 revolute joints and 1 prismatic joint (joint 3) in its initial configuration $\theta = 0$.

(a) Compute the spatial Jacobian J^s and the body Jacobian J^b of the manipulator in the configuration shown.

(b) Now let the robot move so that $\theta_2 = \pi/2$, with all other joints remaining at zero. Compute the spatial Jacobian J^s and body Jacobian J^b in the new configuration.

(c) In which configurations, the one in part (a), or the one in part (b), is the robot in a singular configuration? Justify.

(d) During the execution of a smooth joint trajectory $\theta(t) \in \mathbb{R}^4$, the robot passes through the configuration from part(a) with joint velocities $\dot{\theta}(t) = (0, -1/L, 1, 1/L)$. Find the velocity of the origin of the end effector as seen from the spatial frame at that instant. Note that here we are asking for the velocity of the *point* at the origin of the tool frame, so your answer should just be a vector $\dot{p}_{sb} \in \mathbb{R}^3$.

Problem 2: Singularities of Euler Angles

We have seen previously that the Euler angle representation of a rotation must have a singularity. In this problem we will use our new mathematical tools to prove it!

(a) Let a rotation R be achieved by extrinsic rotations about the spatial z, y, and x axes, i.e. a ZYX Euler angle. Show that the spatial Jacobian for the rotation has a singularity when $\theta_y = \frac{\pi}{2}$.

(b) Prove that a rotation about three arbitrary axes ($R = e^{\hat{\omega}_1\theta_1}e^{\hat{\omega}_2\theta_2}e^{\hat{\omega}_3\theta_3}$) will have a singularity if the three axes are linearly dependent, i.e. $a\omega_1 + b\omega_2 = \omega_3$. (Note that this is the case for e.g. ZYZ Euler angles).

(c) Prove that any rotation about three linearly independent axes will also have a singularity. *Hint: Set $\theta_1 = 0$ and see if you can find a θ_2 that produces a singularity. Note that you can project a vector v onto the plane normal to a second vector w using the equation $v_{proj} = (I - ww^T)v$. Try drawing a diagram!*

Problem 3: Kinematic Singularity: four coplanar revolute joints

Four revolute joint axes with twists $\xi_i = (q_i \times \omega_i, \omega_i),\ i = 1 \cdots 4$ are said to be coplanar if there exists a plane with unit normal n such that:

- Each axis direction is orthogonal to n: $n^T \omega_i = 0$ for all i.
- The vector from q_i to q_j is orthogonal to n: $n^T (q_i - q_j) = 0$ for all i, j.

Show that when four of its revolute joint axes are coplanar, any six degree of freedom manipulator is at a singular configuration.

Problem 4: Manipulability Agility Ability

In robotics, we're often concerned with the question of how easily our robot arms move in space. We know that in certain configurations, we encounter kinematic *singularities*, in which the robot will no longer be able to enjoy its usual range of motion. How can we measure how close we are to encountering one of these singularities? We may use a *manipulability measure*.
One measure of manipulability is the product of the singular values of the spatial jacobian:

$$\mu(\theta) = \prod_{i=1} \sigma_i(\theta) \tag{1}$$

Where σ_i is the i^{th} *singular value* of the robot's spatial jacobian, $J^s(\theta)$. Note that Π means "take the product" in the same way that Σ means "take the sum."
Before we analyze this special function, we'll need a few facts from linear algebra:

1. The null space of a matrix $A \in \mathbb{R}^{m \times n}$ is the set of all nonzero vectors $v \in \mathbb{R}^n$ such that $Av = 0$.
2. For all matrices $A \in \mathbb{R}^{m \times n}$, the null space of A equals the null space of $A^T A$.

Using these facts, let's examine some properties of the manipulability measure!

(a) The singular values σ_i of a matrix $A \in \mathbb{R}^{m \times n}$ are defined as the square roots of the eigenvalues of $A^T A$:

$$\sigma_i = \sqrt{eig(A^T A)} \tag{2}$$

Prove that if the spatial jacobian $J^s(\theta)$ has a singularity, it will have at least one singular value $\sigma_i = 0$. *Hint: when a matrix is singular, at least one of its eigenvalues is zero.*

(b) Prove that the manipulability measure:

$$\mu(\theta) = \prod_{i=1} \sigma_i(\theta) \tag{3}$$

Is equal to zero whenever the Jacobian has a singularity.

(c) Let's graphically interpret the manipulability function using ZYX Euler angles. Letting $\theta_1 = \theta_3 = 0$, find the spatial jacobian of the ZYX Euler angles as a function of θ_2.
Plot the manipulability measure $\mu(\theta_2)$ of this Jacobian as a function of the second Euler angle on the domain $\theta_2 \in [0, 2\pi)$, e.g. using Python. Using your knowledge of Euler angles, interpret the locations of the minima and maxima of $\mu(\theta_2)$.
Hint: Refer to your solution to question 2.