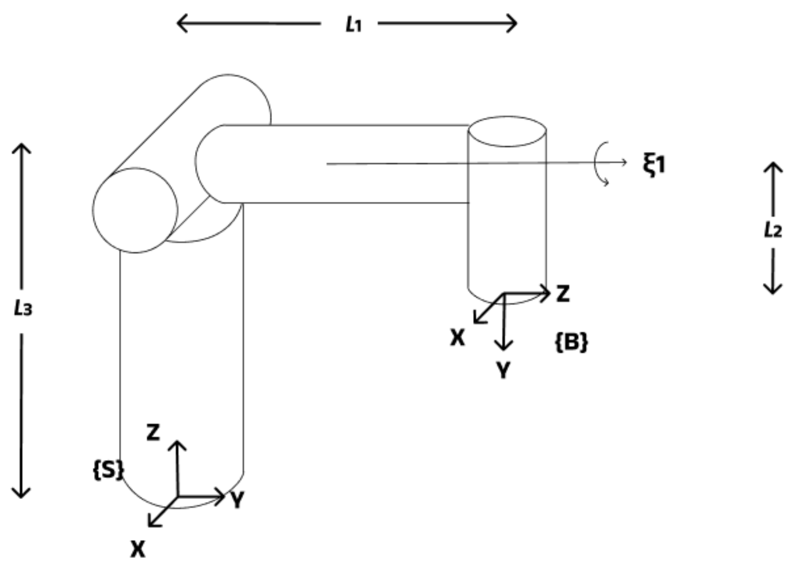
Homework 6: Velocities and Adjoints

EECS/BioE/MechE C106A/206A
Introduction to Robotics

Due: October 19, 2023

Problem 1: Velocities Practice

Find the spatial and body velocity twists for the fixed frame S and the moving frame B in terms of θ, $\dot{\theta}$, and the lengths shown in the diagram. Assume we only consider movement about the ξ_1 axis.



Problem 2: Twists as Velocities

Recall that Chasle's theorem allows us to express any rigid body transform $g \in SE(3)$ as the exponential of a (not necessarily unit) twist $\xi \in \mathfrak{se}(3)$ as $g = e^{\hat{\xi}}$. Consider a trajectory $g(t) \in SE(3)$ that evolves from an initial configuration $g(0)$ to a final configuration $g(1)$ according to the following formula:

$$g(t) = e^{\hat{\xi} t} \cdot g(0) \tag{1}$$

where $t \in [0, 1]$ denotes time. $g(t)$ evolves according to a *constant screw motion*. In this problem, we are going to interpret a twist ξ as a velocity.

(a) Given a desired initial configuration $g_0 \in SE(3)$ and a desired final configuration $g_1 \in SE(3)$, how might we find a twist $\xi \in \mathfrak{se}(3)$ so that a smooth trajectory of the form (1) takes us from $g(0) = g_0$ to $g(1) = g_1$? You do not need to explicitly solve for the twist.

Hint: Your answer should be fairly short. Recall the matrix logarithm.

(b) For $t \in (0, 1)$, find the spatial rigid body velocity $V^s(t)$ of the trajectory you constructed in part (a). Does this velocity depend on time?

(c) For $t \in (0, 1)$, find the body velocity $V^b(t)$ of the trajectory you constructed in part (a). Does this velocity depend on time?

(d) Let a $g \in SE(3)$ be given, with exponential coordinates ξ (a not necessarily unit twist) so that $g = e^{\hat{\xi}}$. Interpret the twist ξ as a rigid body velocity that, when performed uniformly for 1 second, brings a rigid body from the identity configuration to the configuration g. In this way, interpret twists (and the idea of exponential coordinates) in terms of rigid body velocities.

Through this problem, we have shown that a twist in the matrix exponential for homogeneous transformation matrices can be interpreted as a rigid body velocity!

Problem 3: Velocities as Twists

In the previous problem, we showed that we can view twists as velocities, going from the twist exponential representation of rigid body motion to finding the rigid body velocity. Now, we will show the opposite - going from the velocity to finding the twist of our screw motion. With this, we show that both are equivalent.

Consider a smooth rotational trajectory $R(t) \in SO(3)$ where $t \in [0, \infty)$ denotes time. In this problem, we will derive the notion of angular and rigid body velocities directly from our knowledge of exponential coordinates.

(a) Let $t \in [0, \infty)$ and a small $\Delta t > 0$ be given. Argue that there exists $\hat{\omega} \in \mathfrak{so}(3)$ such that

$$R(t + \Delta t) = e^{\hat{\omega}\Delta t} \cdot R(t) \tag{2}$$

Note that ω is a function of both t and Δt.

(b) Now take the limit as $\Delta t \to 0$. Show that in this limit, $\hat{\omega}$ approaches $\dot{R}R^T = \hat{\omega}^s(t)$. That is, in the limit, this infinitessimal rotation approaches the spatial angular velocity of R. *Hint: It may help to recall that for small Δt we have $e^{A\Delta t} \approx I + A\Delta t$. It may also help to recall the limit definition of the derivative.*

(c) Conclude that the spatial angular velocity of R is simply the *instantaneous rotation axis* of the body, with magnitude equal to the instantaneous angular speed.

(d) Repeat the exercise in parts (a)-(c) except with a smooth *rigid-body motion* trajectory $g(t) \in SE(3)$. Interpret the spatial velocity $V^s(t)$ in terms of the twist associated with the *instantaneous screw motion* that the body is undergoing at time t.

Problem 4: Properties of the Adjoint

(a) Show that $(\mathrm{Ad}_g)^{-1} = \mathrm{Ad}_{g^{-1}}$ for all $g \in SE(3)$.

(b) Show that $\mathrm{Ad}_{g_1 g_2} = \mathrm{Ad}_{g_1} \mathrm{Ad}_{g_2}$ for all $g_1, g_2 \in SE(3)$.