Homework 2: Exponential Coordinates

EECS/ME/BioE C106A/206A Introduction to Robotics

Fall 2023

Note This problem set includes two programming components. Your deliverables for this assignment are:

1. A PDF file submitted to the `HW2 (pdf)` Gradescope assignment with all your work and solutions to the written problems.
2. The provided `kin_func_skeleton.py` and `hw2.py` file submitted to the `HW2 (code)` Gradescope assignment with your implementation to the programming components. Make sure to select both files when submitting your assignment.

Theory

0.1 Rigid Body Transformations

In this class, we work with rigid body transformations. This means whatever object we're translating or rotating (such as a robot arm) retains its physical structure. It just moves around. We showed that rigid body transformations have 2 critical components:

1. Length preservation: $\forall p, q \in \mathbb{R}^3$, $||p - q|| = ||G(p) - G(q)||$
2. Orientation preservation: $\forall$ vectors $v, w \in \mathbb{R}^3, G(v \times w) = G(v) \times G(w)$

0.2 Rotations

0.2.1 Rotations Review

Last week, we covered the idea of rotations and rotation matrices. This is a rigid body transformation that doesn't involve any translation. We saw that we can represent rotations as matrices: R_{AB}. There are a few interpretations of this rotation matrix:

- R_{AB} is composed of 3 unit column vectors that represent the B frame in terms of the A frame (x_{AB}, y_{AB}, z_{AB}).
- R_{AB} applied on some point q in the B frame will tell us what that point would be in the A frame: $q_A = R_{AB} q_B$
- If we originally had some point in the standard coordinate frame and then we rotated it, we can find the new location of that point in the standard coordinate frame using the rotation matrix: $q' = R_{AB} q$

You should make sure you understand why all of these interpretations are correct and equivalent.

0.2.2 SO(3)

Normally, we don't just have a single transformation we want to show. Instead, we want to find a transformation matrix as a function of time. In order to parameterize our motion by time, we can use matrix exponentials to generate our transformation matrices. The proof of these is covered in lecture and discussion.

In order to create our rotation matrix, we have the following formula:

$$R(t) = e^{\hat{\omega} t}$$

where ω is our axis of rotation, $\hat{\omega}$ is a skew-symmetric matrix generated from ω, and t is the extent of rotation (often written as θ). The axis of rotation ω doesn't have to be one of the standard coordinate axes. You can look at Problem 3 for the equations on solving out the exponential and finding the rotation matrix (the Rodrigues' Formula).

The skew-symmetric matrix $\hat{\omega}$ is in the $so(3)$ group, whereas rotation matrices R are in the $SO(3)$ group.

0.3 Homogeneous Coordinates

Now, we're going to take a leap forward and incorporate translation! In order to do so, we first develop the idea of **homogeneous coordinates**. This adds a 4th dimension that allows us to differentiate between points and vectors.

Specifically, a point p would have a 1 added in the 4th row, and a vector $\vec{v}$ would have a 0 added:

$$p = \begin{bmatrix} p_x \\ p_y \\ p_z \\ 1 \end{bmatrix} \qquad \vec{v} = \begin{bmatrix} v_x \\ v_y \\ v_z \\ 0 \end{bmatrix}$$

0.4 Homogeneous Transformation Matrix

Now that we are armed with homogeneous coordinates, we can develop a single matrix representation for full rigid body transformations, incorporating both rotation and translation!

If our rigid body is rotated by the rotation matrix R_{AB} and translated by an XYZ translation vector t_{AB}, our 4x4 homogeneous transformation matrix takes the following form:

$$G_{AB} = \begin{bmatrix} R_{AB} & t_{AB} \\ 0 & 1 \end{bmatrix}$$

We can compute transformations in the same way as rotations: $q_A = G_{AB} q_B$.

We can also stack and invert these:

$$\begin{gathered} G_{AC} = G_{AB} G_{BC} \\ G_{AC} = G_{CA}^{-1} \end{gathered} \qquad G^{-1} = \begin{bmatrix} R^T & -R^T t \\ 0 & 1 \end{bmatrix}$$

0.5 Exponential Coordinates

We've already figured out how to parameterize our rotation matrices by time using a matrix exponential. Let's now do the same with our homogeneous transformation matrices! In order to do so, we're going to bring in the idea of **exponential coordinates**.

Exponential coordinates are composed of 2 parts: ξ and θ.

ξ (the Greek symbol "xi") is a 6x1 vector known as a twist, and it describes the *way* we are transforming. θ is a scalar that describes the *extent* to which the transformation occurs.

0.5.1 Twists

A twist $\xi \in \mathbb{R}^6$ incorporates linear and angular velocity components (to understand why we refer to this as "velocity," look at the proof for exponential coordinates). It is composed of two 3x1 vectors, v and ω.

$$\xi = \begin{bmatrix} v \\ \omega \end{bmatrix} \in \mathbb{R}^6$$

0.5.2 Pure Rotation (revolute joints)

If we are simply rotating our rigid body, we consider the motion a revolute joint. If ω is our axis of rotation and q is some point on that axis, we calculate our twist in the following way:

$$\xi = \begin{bmatrix} -\omega \times q \\ \omega \end{bmatrix}$$

Our velocity v is equal to $-\omega \times q$.

0.5.3 Pure Translation (prismatic joints)

If we are simply translating our body (no rotation), we consider the transformation to be a prismatic joint. If our velocity is v, our twist is calculated as

$$\xi = \begin{bmatrix} v \\ 0 \end{bmatrix}$$

Our axis of rotation ω is 0.

0.5.4 Translation and Rotation (screw joint)

Sometimes, we have both translation and rotation! This is called a screw joint. **Every rigid body transformation can be represented as a single screw motion.** Our twist for this takes the following form:

$$\xi = \begin{bmatrix} -\omega \times q + h\omega \\ \omega \end{bmatrix}$$

h is the ratio of the amount you translate to the amount you rotate. ($h = 0$ corresponds to pure rotation.)

0.5.5 Exponential Coordinates

Our exponential coordinates for the homogeneous transformation matrix will be (ξ, θ). You'll need to calculate both the way the transformation happens and the extent of the transformation.

We can calculate our homogeneous transformation matrix as follows:

$$G = e^{\hat{\xi}\theta}$$

We can also apply this to a point:

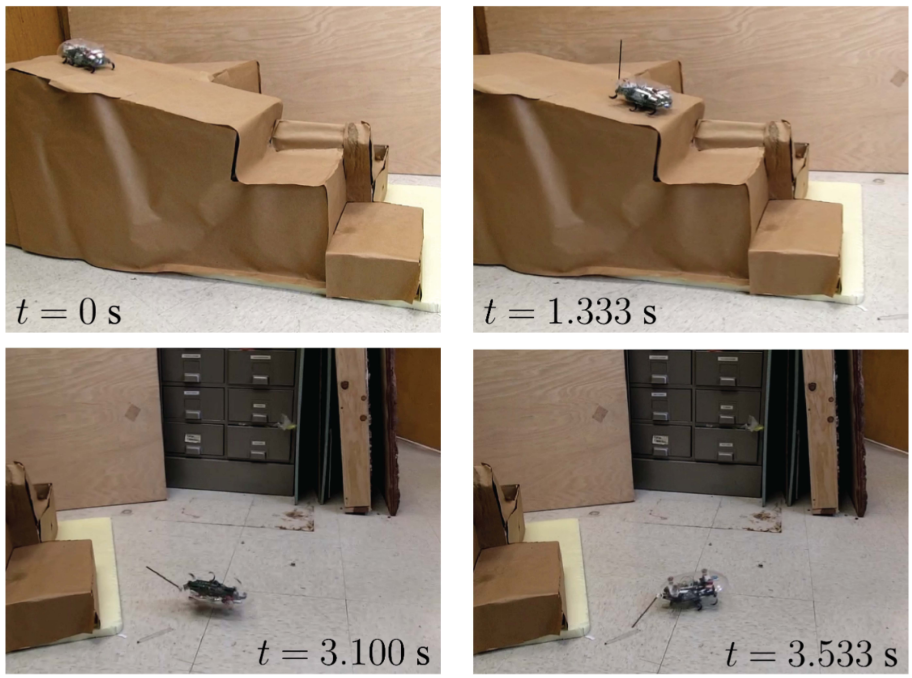
$$p(t) = e^{\hat{\xi}t}p(0)$$

Refer to Problem 3 to understand how this exponentiation is carried out!

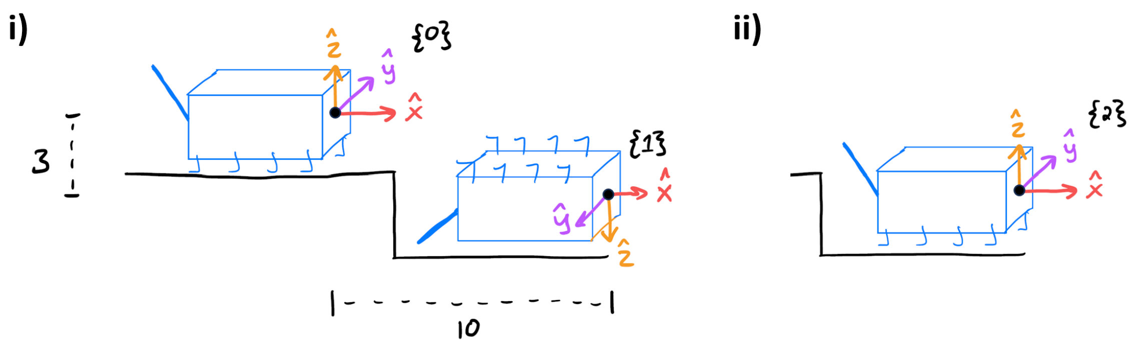
Our hat map $\hat{\xi}$ is in the se(3) group. Our homogeneous transformation matrix G_{AB} is in the SE(3) group.

Problem 1: Running VelociROACH

The Berkeley VelociROACH robot is able to recover from flipping onto its back by using its tail to right itself! https://www.youtube.com/watch?v=h9pN1OF5nlU In this problem you will calculate the transformations of the VelociROACH's body as it runs down some stairs.



The VelociROACH robot enjoying a nice stroll.



(i) The VelociROACH runs down a step and falls over in frame {1}.
(ii) It uses its tail to return to its original orientation. (Note that the origins of frames {1} and {2} are located at the same position.)

(a) Recall that the 4x4 homogeneous *transformation matrix* expresses both the rotation and translation of a rigid body. Find the transformation matrices T_{02} of frame $\{2\}$ relative to frame $\{0\}$ and T_{01} of frame $\{1\}$ relative to frame $\{0\}$. (Assume that the y-position of the VelociROACH is the same across all frames.)

(b) Show how to find T_{21} in terms of T_{01} and T_{02} and verify that it has no translation component.

(c) In frame $\{1\}$, the tip of the robot's tail lies at point (-7, 0.5, 0.5). Use the appropriate transformation matrix to find its position relative to frame $\{0\}$.

(d) Recall that all rigid-body transforms can also be expressed as a *screw motion*, i.e. a rotation and translation of θ about a fixed screw axis ξ. Find the screw axis and θ corresponding to T_{02} (pure translation) and T_{21} (pure rotation).

(e) Use the result from the previous part to write the exponential coordinates of both transforms.

Problem 2: Exponential Coordinates for Rotations

Recall that for any rotation matrix $R \in SO(3)$, there exists a unit axis vector $\omega \in \mathbb{R}^3$, a corresponding skew symmetric matrix $\hat{\omega} \in \mathfrak{so}(3)$, and a scalar θ such that $R = e^{\hat{\omega}\theta}$. Together, ω and θ are the *exponential coordinates* of the rotation. Geometrically, they parameterize a rotation $R = e^{\hat{\omega}\theta}$ such that Rv moves a vector $v \in \mathbb{R}^3$ by θ radians about the unit axis ω. (Also, while it is not necessary for this problem, recall that the exponential is derived from solving a differential equation relating the angular velocity of a point and the axis: $v = \dot{q} = \omega \times q(t)$, $q(t) = R = e^{\hat{\omega}t}q_0$)

(a) Let $\omega = (\omega_1, \omega_2, \omega_3)^T \in \mathbb{R}^3$ be a unit vector and recall that we define the hat operator as

$$\hat{\omega} = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix} \tag{1}$$

Note that we denote this operator as either $\hat{\omega}$ or $\omega^\wedge$ interchangeably. Further, we define the "vee" operator $^\vee$ as the inverse of hat, so that $\hat{\omega}^\vee = \omega$. "vee" is defined on $\mathfrak{so}(3)$ and returns a 3-vector.

Let $\theta \in [0, \pi]$ be a scalar. Show that the matrix $\hat{\omega}\theta$ has eigenvalues $\{0, i\theta, -i\theta\}$.

(b) Let R be the rotation matrix for which (ω, θ) is a set of exponential coordinates. i.e. $R = e^{\hat{\omega}\theta}$. Find the eigenvalues of R.

Hint: Recall the properties of the matrix exponential we introduced in Homework 0.

(c) For what value(s) of the rotation angle θ does R have a single distinct *real* eigenvalue? What about 2 distinct real eigenvalues? Can it ever have 3?

Hint: Recall Euler's formula.

(d) Interpret your answer to part (c) geometrically. When R has exactly 1 real eigenvalue, what is it and what is the corresponding eigenvector? Why does this make sense geometrically given that R is a rotation matrix? What about when R has two distinct real eigenvalues? You should answer this question without ever carrying out a direct eigenvector computation.

(e) Show that for *any* $\omega \in \mathbb{R}^3$:

$$\hat{\omega}^T\hat{\omega} = (\omega^T\omega)I - \omega\omega^T \tag{2}$$

Hints: Apply both sides to a vector x - if $Ax = Bx$ for all x, then $A = B$. Remember the vector triple product!

(f) Let's see how we can extract the magnitude of angular velocity from $\hat{\omega}$. Show that the following identity holds, where $\mathrm{tr}(\cdot)$ is the sum of the diagonal entries of a matrix.

$$||\omega||_2^2 = \frac{1}{2}\mathrm{tr}(\hat{\omega}^T\hat{\omega}) \tag{3}$$

Hint: How are $\mathrm{tr}(xx^T)$ and x^Tx related for a vector x? Note: This relates the magnitude of ω to the Frobenius norm of $\hat{\omega}$!

Problem 3: Implementing Exponential Coordinates

What good is all this theory if we can't use it for something? In order to see the applications of the exponential map, we'll first need to implement a few fundamental equations in code. Fill in the provided `kin_func_skeleton.py` file to implement the following formulas using numpy. Test your implementation with the provided test cases by simply running `python kin_func_skeleton.py` in the command line. You will need this code to start Lab 3.

(a) The "hat" $(\cdot)^\wedge$ operator for rotation axes in 3D.

- Input: 3×1 vector, $\omega = [\omega_x, \omega_y, \omega_z]^T$
- Output: 3×3 matrix,

$$\hat{\omega} = \begin{bmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{bmatrix} \tag{4}$$

(b) Rotation matrix in 3D as a function of ω and θ

- Input: 3×1 vector, $\omega = [\omega_x, \omega_y, \omega_z]^T$ and scalar, θ
- Output: 3×3 matrix following the Rodrigues' Formula:

$$R(\omega, \theta) = e^{\hat{\omega}\theta} = I + \frac{\hat{\omega}}{\|\omega\|}\sin(\|\omega\|\theta) + \frac{\hat{\omega}^2}{\|\omega\|^2}(1 - \cos(\|\omega\|\theta)) \tag{5}$$

(c) The "hat" $(\cdot)^\wedge$ operator for Twists in 3D.

- Input: 6×1 vector, $\xi = \left[v^T, w^T\right]^T = [v_x, v_y, v_z, \omega_x, \omega_y, \omega_z]^T$
- Output: 4×4 matrix,

$$\hat{\xi} = \begin{bmatrix} \hat{\omega} & v \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -\omega_z & \omega_y & v_x \\ \omega_z & 0 & -\omega_x & v_y \\ -\omega_y & \omega_x & 0 & v_z \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{6}$$

(d) Homogeneous transformation in 3D as a function of twist ξ and joint angle θ.

- Input: 6×1 vector, $\xi = \left[v^T, w^T\right]^T = [v_x, v_y, v_z, \omega_x, \omega_y, \omega_z]^T$ and scalar θ
- Output: 4×4 matrix,

$$g(\xi, \theta) = e^{\hat{\xi}\theta} = \begin{cases} \begin{bmatrix} I & v\theta \\ 0 & 1 \end{bmatrix} & w = 0 \\ \\ \begin{bmatrix} e^{\hat{\omega}\theta} & \frac{1}{\|w\|^2}\left(\left(I - e^{\hat{\omega}\theta}\right)(\hat{\omega}v) + \omega\omega^T v\theta\right) \\ 0 & 1 \end{bmatrix} & \omega \neq 0 \end{cases} \tag{7}$$

(e) Product of exponentials in 3D.

- Input: n 6D vectors, $\xi_1, \xi_2, \ldots, \xi_n$ and scalars, $\theta_1, \theta_2, \ldots, \theta_n$
- Output:

$$g(\xi_1, \theta_1, \xi_2, \theta_2, \ldots, \xi_n, \theta_n) = e^{\hat{\xi}_1\theta_1} e^{\hat{\xi}_2\theta_2} \ldots e^{\hat{\xi}_n\theta_n} \tag{8}$$

Problem 4: Satellite System

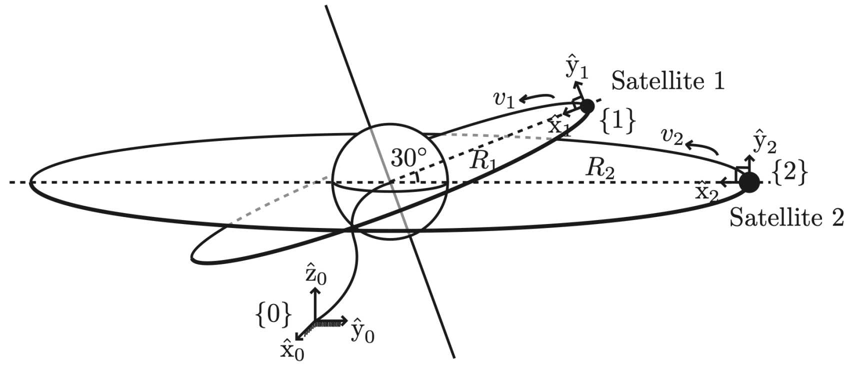


Figure 1: Two satellites circling the Earth. In both cases, the satellite's z-axis points directly into the page (tangent to the orbit).

Two satellites are circling the Earth as shown in Figure 1. Frames $\{1\}$ and $\{2\}$ are rigidly attached to the satellites in such a way that their $\hat{x}$-axes always point toward the Earth. Satellite 1 moves at a constant speed v_1, while satellite 2 moves at a constant speed v_2. To simplify matters, ignore the rotation of the Earth about its own axis. The fixed frame $\{0\}$ is located at the center of the Earth. Figure 1 shows the position of the two satellites at $t = 0$. For the following questions, you may leave your answers in terms of the products of matrices you have calculated in previous parts.

(a) Derive the homogeneous transformation matrix for satellite 2, T_{02}, at time $t = 0$.

(b) Now find the transformation matrix for any time t, $T_{02}(t)$. *Hint: See if you can determine the time-dependent transform from frame 2's configuration at time t to its initial configuration, and then apply part (a)*

(c) Find the twist ξ_2 for the motion of satellite 2. ξ_2 should satisfy

$$T_{02}(t) = e^{\hat{\xi}_2 t} T_{02}(0)$$

Hint: You should not have to take any matrix logarithms here. Think about what each element of ξ represents.

(d) Now, let's examine satellite 1. Find the transformation matrix for satellite 1 as a function of time, $T_{01}(t)$. *Hint: Does the motion of satellite 1 looks similar to that of satellite 2? How are they different?*

(e) Using your results from part (b) and (d), find $T_{21}(t)$.

(f) Find the twist ξ_1 such that ξ_1 satisfies

$$T_{01}(t) = e^{\hat{\xi}_1 t} T_{01}(0)$$

(g) Fill in the corresponding parts of `hw2.py` to implement your answers to parts (a)-(f) above. Note that your credit for this problem will be awarded by the autograder configured to the `HW2 (code)` assignment on Gradescope. Make sure you submit both `hw2.py` and `kin_func_skeleton.py`!

You can visualize the motion of these frames by running the `g_t_vis.py` and `xi_vis.py` after completing the relevant sections of `hw2.py`. Note that both `g_t_vis.py` and `xi_vis.py` will only work after filling out `kin_func_skeleton.py`, and `xi_vis.py` needs all parts of `hw2.py` completed. Use your scroll wheel to zoom camera, ctrl+drag to rotate camera, and shift+drag to pan camera. This may be useful for verifying your computations before submitting to Gradescope, and fun to play with as well. What cool rigid body motions can you come up with?