EECS106A Discussion 9: Lagrangian dynamics in 3D and state spaces

1 Review: Lagrangian dynamics

As we saw last week, the kinetic energy of a rigid body can be expressed as:

$$T = \frac{1}{2}(\omega^b)^T \mathcal{I} \omega^b + \frac{1}{2} m v^T v \tag{1}$$

$$= \frac{1}{2}(V^b)^T \mathcal{M} V^b \tag{2}$$

where we obtain the second equation by grouping the linear velocity v and body angular velocity ω^b into the body twist V^b, and the *inertia tensor* $\mathcal{I}$ and mass m into the *generalized inertia matrix*:

$$\mathcal{M} = \begin{bmatrix} mI_3 & 0 \\ 0 & \mathcal{I} \end{bmatrix}$$

we can then write our Lagrangian as

$$L = T - V \tag{3}$$

and our equations of motion as

$$\Upsilon = \frac{d}{dt}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} \tag{4}$$

$$= M(q)\ddot{q} + C(q, \dot{q})\dot{q} + G(q). \tag{5}$$

$M(q)$, the matrix that multiplies the acceleration terms $\ddot{q}$, is called the *inertia matrix* or the *manipulator inertia matrix* in the context of a robotic manipulator. Note that we have now covered three different uses of the word "inertia" (sorry!):

1. The inertia tensor $\mathcal{I} \in \mathbb{R}^{3\times 3}$
2. The generalized inertia matrix $\mathcal{M} \in \mathbb{R}^{6\times 6}$
3. The (manipulator) inertia matrix $M(q) \in \mathbb{R}^{n\times n}$ for n generalized coordinates

We will discuss the inertia tensor further in the next section. For a system composed of multiple links or parts, we have $T = \sum_i T_i$ and $V = \sum_i V_i$.

Problem 1: *How does equation (1) simplify if the rigid body is a point mass?*
For a point mass, there is no rotational kinetic energy (the inertia tensor is zero). This is because there's only a single point on a point mass, and spinning the point mass doesn't cause it to move. (Compare to problem 2e on HW8 - if the rigid body has length $l = 0$, what happens to the inertia?). Therefore, equation (1) simplifies to

$$T = \frac{1}{2} m v^T v$$

Problem 2: *How does equation (1) simplify if motion is limited to a 2D plane?*
When motion is limited to a 2D plane, there is only one axis of rotation, so we can express the rotational velocity as a scalar $\dot{\theta}$. This means that $\mathcal{I}$ is also a scalar, and equation (1) simplifies to

$$T = \frac{1}{2}I\dot{\theta}^2 + \frac{1}{2}mv^T v$$

as we saw in discussion 8 last week.

2 Interpreting 3D inertia

2.1 Inertia tensor of a box

Consider the simple rigid body shown in figure 1:

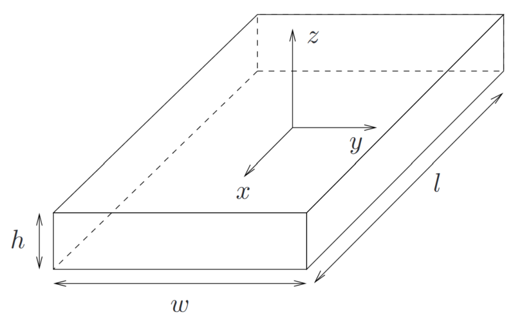

Figure 1: A box with evenly distributed mass.

Let's place the body reference frame at the center of mass of the box, and assume that the box is homogeneous (its mass is distributed evenly). If we rotate the box around some axis ω, we can see that the distribution of mass about the axis will depend on the orientation of ω.

Problem 3: *Let $l > w > h$, and consider rotations about the x, y, and z axes. Which rotation should have the highest inertia? Which should have the lowest?*
We know that the rotational inertia is just due to the velocity of mass in the body induced by the rotational motion, and that points further from the axis of rotation have larger velocities. Since we assume the box is homogeneous, the further a point mass is away from the axis of rotation, the larger the inertia. Therefore expect the rotation about x to have the lowest inertia, since points are up to $w/2$ away from the axis of rotation. Similarly, rotation about z should have the highest inertia.

By calculating the velocity of each point in the box due to a rotation and integrating over the mass density times the velocity at each point, it's possible to calculate that the inertia tensor of this box is given by

$$\mathcal{I} = \begin{bmatrix} \frac{m}{12}(w^2+h^2) & 0 & 0 \\ 0 & \frac{m}{12}(l^2+h^2) & 0 \\ 0 & 0 & \frac{m}{12}(l^2+w^2) \end{bmatrix} \tag{6}$$

Problem 4: *You've successfully programmed your robot arm to toss the box shown above straight up in the air with speed v_z and a fancy spin given by $\omega = [\omega_x, \omega_y, \omega_z]^T$. Let the height of the hand be given by r. What is the Lagrangian for the box at the moment it leaves the robot's hand?* We have

$$\begin{aligned} L &= T - V \\ &= \frac{1}{2}(\omega^T \mathcal{I} \omega + m v^T v) - V \\ &= \frac{m}{24}(\omega_x^2(w^2+h^2) + \omega_y^2(l^2+h^2) + \omega_z^2(l^2+w^2)) + \frac{1}{2}v_z^2 m - mgr \end{aligned}$$

2.2 Inertia matrix of a manipulator

You and a friend are having a dance competition but need an impartial judge, so you decide to use a robot! To indicate the victor, the arm shown in figure 2 needs to be able to wave in their direction. The links of the robot have masses m_1 and m_2, lengths l_1 and l_2, and inertia tensors given by $\mathcal{I}_i = \begin{bmatrix} I_{x_i} & 0 & 0 \\ 0 & I_{y_i} & 0 \\ 0 & 0 & I_{z_i} \end{bmatrix}$.

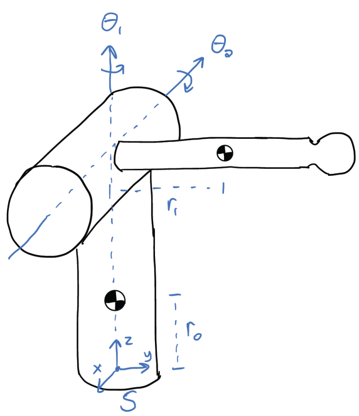

Figure 2: A simple robojudge in its zero configuration.

Problem 5: *Find the inertia matrix and Lagrangian for this system as a function of the joint angles θ_1 and θ_2. How can we interpret the inertia matrix?* To find the kinetic energy of our 2-link robot, we start from equation 2 and replace the body twist with the Jacobian, which allows us to express the kinetic energy as a function of an arbitrary robot configuration θ:

$$\begin{aligned} T &= \frac{1}{2}(V^b)^T \mathcal{M} V^b \\ &= \frac{1}{2}\sum_i (V_i^b)^T \mathcal{M}_i V_i^b \\ &= \frac{1}{2}\sum_i (J_i^b \dot{\theta})^T \mathcal{M}_i J_i^b \dot{\theta} \\ &= \frac{1}{2}\dot{\theta} M(\theta) \dot{\theta} \end{aligned}$$

where V_i^b and J_i^b are defined relative to a body frame located at the center of mass for each link, and for this two-link system $M(\theta)$ is

$$M(\theta) = (J_1^b)^T \mathcal{M} J_1^b + (J_2^b)^T \mathcal{M} J_2^b$$

.
To calculate J_1^b and J_2^b, we need to calculate the twist for both links relative to the center of mass of each link. The jth column of J_i^b is the twist of joint j relative to the center of mass frame of joint i if $j \leq i$. If $j > i$, the column is a 0 vector because joints that come after i in the chain do not affect its velocity. Using the geometry of the manipulator, we find that

$$J_1^b = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{bmatrix} \quad J_2^b = \begin{bmatrix} -r_1 \cos(\theta_2) & 0 \\ 0 & 0 \\ 0 & -r_1 \\ 0 & -1 \\ -\sin(\theta_2) & 0 \\ \cos(\theta_2) & 0 \end{bmatrix}$$

Multiplying everything out (e.g. using Sympy), we get

$$M(\theta) = \begin{bmatrix} I_{z_1} + I_{y_2} \sin^2(\theta_2) + I_{z_2} \cos^2(\theta_2) + m_2 r_1 \cos^2(\theta_2) & 0 \\ 0 & I_{x_2} + m_2 r_1^2 \end{bmatrix}$$

Because this is an open-chain manipulator and we chose the joint angles as our generalized coordinates, $M(\theta)$ is the inertia matrix of our final Lagrangian dynamics. This means we can intuitively interpret its elements as the amount of external force we need to apply to accelerate each joint by a unit amount (very useful for control!).
To get the final Lagrangian, the only thing left to do is calculate the potential energy. Letting the height of link 1 be l_1, we have

$$V = m_1 g h_1 + m_2 g h_2 = g(m_1 r_0 + m_2(l_1 - r_1 \sin(\theta_2)))$$

and $L = T - V$.

3 Intro to linear dynamics

When working with Lagrangian dynamics, we introduced the "generalized coordinates," which describe the motion of a system in terms of a set of n scalar coordinates q_i. In control theory, we call these coordinates the *state* of our system, and the set of all possible values of the coordinates form the *state space*. Along the same line, choosing an appropriate set of coordinates is often referred to as *state space modelling*.

Problem 6: *If we want to model the motion of the box being tossed from the previous section, what state space could we choose? What if we want to model an arbitrary toss in any direction and with any spin?*
For the model from the previous section, we only have translational motion in the z direction and rotational motion about a single axis. Therefore, we can write $x = \begin{bmatrix} z & \dot{z} & \theta & \dot{\theta} \end{bmatrix}^T$ where we choose the coordinates so that θ expresses the angle of rotation about the axis ω. Note that we include both the coordinates and the first derivatives because we want the acceleration terms to show up in our linear dynamics below. For the general case, we add the variables x, y, ϕ, and ψ and their derivatives to express general rotation and translation.

The easiest kind of systems to analyze are *linear systems*, which have the form

$$\dot{x}(t) = A(t)x(t) + B(t)u(t) \tag{7}$$

$$y(t) = C(t)x(t) + D(t)u(t) \tag{8}$$

where $x \in \mathbb{R}^n$ is the state vector, $u \in \mathbb{R}^m$ is a set of control inputs, $A \in \mathbb{R}^{n \times n}$ describes how the system state will change in the absence of a control input, and $B \in \mathbb{R}^{n \times m}$ describes how the control input affects the system state. If A, B, C, and D do not change over time, we say that the system is *linear time-invariant* or LTI.

In general, we may not be able to directly measure or observe all of the system states, so the observation $y \in \mathbb{R}^{n_o}$ is some linear function of x and u. In the case where we can observe all of x directly, we take $D = 0$, $C = I$ and $y(t) = x(t)$.

Unfortunately, most interesting systems do not have linear dynamics. However, it turns out that nonlinear dynamics can often be approximated with good accuracy in the local neighborhood of a point by "linearizing" them about this point. This means that analyzing the behavior and control of linear systems is still extremely useful, especially since it's very difficult to analyze nonlinear systems.