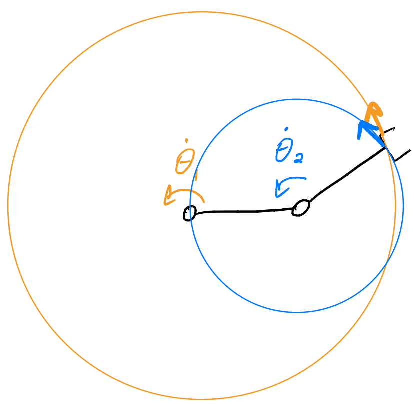
Discussion 7: Jacobians and singularities

2 DoF Spatial Jacobian



Let $\dot\theta_2 = 0 \Rightarrow V^s = \xi_1 \dot\theta_1$

recall: $V^s_{st} = Ad_{g_{st}} V^b_{st}$, $\xi^s = Ad_{g_{st}} \xi^b$

let $\dot\theta_1 = 0 \Rightarrow V^s = Ad_{e^{\hat\xi_1\theta_1}} \xi_2 \dot\theta_2$

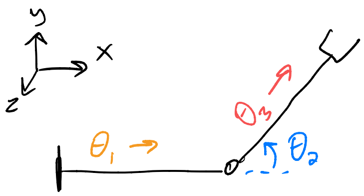
General case: just add them together!

$$V^s = \xi_1\dot\theta_1 + \underbrace{Ad_{e^{\hat\xi_1\theta_1}} \xi_2}_{\xi_2'} \dot\theta_2$$

$$= \underbrace{\begin{bmatrix} \xi_1 & \xi_2' \end{bmatrix}}_{J^s(\theta)} \underbrace{\begin{bmatrix} \dot\theta_1 \\ \dot\theta_2 \end{bmatrix}}_{\dot\theta}$$

For n joints: $\underbrace{V^s}_{6\times 1} = \underbrace{J^s(\theta)}_{6\times n} \underbrace{\dot\theta}_{n\times 1}$

Singularities



$$\omega_1 = \omega_3 = \omega_3' = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{(prismatic joints)} \qquad v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

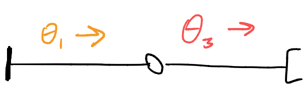
fully collapsible robot arm

$$\omega_2 = \omega_2' = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \qquad v_2' = -\omega_2' \times \begin{bmatrix} \theta_1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ -\theta_1 \\ 0 \end{bmatrix}$$

$$v_3' = \begin{bmatrix} \cos\theta_2 \\ \sin\theta_2 \\ 0 \end{bmatrix} \qquad J^s(\theta) = \begin{bmatrix} 1 & 0 & \cos\theta_2 \\ 0 & -\theta_1 & \sin\theta_2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

Since $V^s = J^s(\theta)\dot{\theta}$, $J^s(\theta)$ provides info about the set of possible velocities of the end effector. Here, these 0's tell us that the end effector will never have translation in the z-direction or rotation about the x- or y-axes. This makes sense for a planar robot!

Now, let $\theta_2 = 0$: $\xi_3' = \xi_1 \Rightarrow J^s(\theta) = \begin{bmatrix} 1 & 0 & 1 \\ 0 & -\theta_1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$

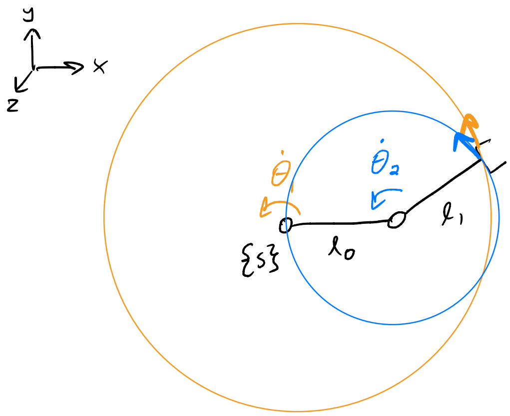


The 2 prismatic joints point in the same direction $\Rightarrow$ lose a degree of freedom!

This corresponds to a loss of rank of $J^s(\theta)$: the span of output velocities has collapsed from 3 dimensions (translation in the xy-plane and rotation about the z-axis) to 2 (translation in the x-direction and coupled (rotation about z-axis, translation in y-direction))

Bonus example

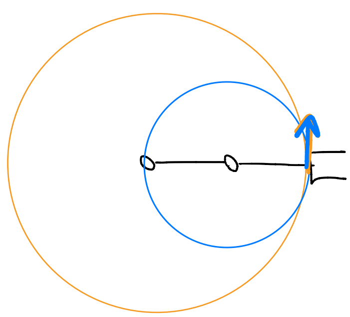
Returning to the 2 DoF manipulator:



$$\omega_1 = \omega_2' = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \qquad v_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$v_2' = -\omega_2' \times \begin{bmatrix} l_0 \cos\theta_1 \\ l_0 \sin\theta_1 \\ 0 \end{bmatrix} = \begin{bmatrix} l_0 \sin\theta_1 \\ -l_0 \cos\theta_1 \\ 0 \end{bmatrix}$$

Now, let $\theta_2 = 0$:



Visually, it looks like we have lost a DoF in the output velocity space. But if we check the Jacobian, there's actually no dependence on θ_2!

$$J^s(\theta) = \begin{bmatrix} 0 & l_0 \sin\theta_1 \\ 0 & -l_0\cos\theta_1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 1 \end{bmatrix}$$

We can see that no matter the configuration of this arm, $J^s(\theta)$ will always have rank 2. This is because even though both joints induce angular velocity, one of them also induces linear velocity, so between rotation and translation of the end effector we have 2 DoF.

However, even though this configuration is not singular for 2D/3D twists, it does prevent translation in the x-direction. We can alternatively define a Jacobian for 2D Cartesian velocities w/o rotation:

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = J^c(\theta) \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix}$$

For this robot, from geometry, we have $J^c(\theta) = \begin{bmatrix} -l_0 \sin\theta_1 - l_1 \sin(\theta_1+\theta_2), & -l_1 \sin(\theta_1+\theta_2) \\ l_0\cos\theta_1 + l_1 \cos(\theta_1+\theta_2) & l_1\cos(\theta_1+\theta_2) \end{bmatrix}$

and when $\theta_2 = 0$, $J^c(\theta)$ has rank 1 as expected!

For this class, you only need to worry about the spatial/body twist Jacobians though!