EECS C106A Discussion 6: Velocities and Adjoints

1 Representing Velocities

We have spoken about rigid body motion so far as a transformation between two points. The g matrix relates the positions of two coordinate frames to one another. Often, we also want to talk about the velocity, or the rate of change of the position of some point with respect to a given reference frame.

When dealing with robots, there are two reference frames that are relevant: the spatial and the body frames.

- Let's fix frame A and consider it our spatial frame (this might be our world frame).
- Let's say frame B is moving - we'll call this our body frame (this might be one of our robot arms).
- Finally, let's say we have some point q that's attached to B (this might be the tip of a gripper attached to the arm).

Let's try to find the velocity of the arbitrary point q. Your intuition should tell you that the velocity of the point q with respect to frame B is just 0. Since q and B are moving together, they have no velocity with respect to each other. There are, however, other ways in which we can express the motion of q.

The velocity is the rate of change of its position with *with respect to a reference frame*. Thus, if we have a frame called U, the velocity of the point q with respect to U is $\dot{q}_u(t)$, where $q_u(t)$ is the point's position with respect to U as a function of time. Now we have a time-dependent velocity vector $\dot{q}_u(t)$ in frame U.

However, this velocity vector can be viewed from any other frame V, and can therefore be expressed from V. Recall that we can change the frame of reference if we have transformation g_{uv}. When representing velocities, it's important to keep in mind that there are two (often different) frames that are relevant.

2 Rigid Body Velocities

Let's say we have a fixed frame A and a moving frame B. By construction, let's also have point q that's attached to frame B. We call A the *spatial* coordinate frame, and B the *body* coordinate frame.

2.1 Spatial Velocity

Since frame B is moving, the transformation between A and B is time-dependent:

$$g_{ab}(t) = \begin{bmatrix} R_{ab}(t) & p_{ab}(t) \\ 0 & 1 \end{bmatrix} \tag{1}$$

Because q is fixed to frame B, its coordinates with respect to B, q_b, are constant. Its coordinates in frame A, however, are time-dependent:

$$q_a(t) = g_{ab}(t)q_b \tag{2}$$

Problem: Differentiate the position $q_a(t)$ with respect to time to find its spatial velocity.
Now, the velocity of this point with respect to A, and also viewed from A, is found via differentiation:

$$\dot{q}_a(t) = \dot{g}_{ab}(t)q_b \tag{3}$$

Following the textbook, we define this quantity to be $v_{q_a}(t) := \dot{q}_a(t)$. We can further express q_b in terms of $q_a(t)$ by the appropriate transformation:

$$\boxed{v_{q_a}(t) := \dot{q}_a(t) = \dot{g}_{ab}(t)q_b = \underbrace{\dot{g}_{ab}(t)g_{ab}^{-1}(t)}_{:=\widehat{V}_{ab}^s} q_a = \widehat{V}_{ab}^s q_a} \tag{4}$$

It turns out that $\boldsymbol{\dot{g}_{ab}(t)g_{ab}^{-1}(t)}$ is a hat map, and we define it to be the *spatial velocity* $\boldsymbol{\widehat{V}_{ab}^s}$. Notice that the spatial velocity is a twist, and the full expression $\dot{q}_a(t) = \dot{g}_{ab}(t)g_{ab}^{-1}(t)q_a$ is a differential equation. It turns out that twists can be interpreted as velocities and that velocities can be interpreted as twists! You will explore this further in your homework.

Problem: Find the spatial velocity twist coordinates.

Going forward with the algebra and dropping notation for time-dependence, we have

$$\begin{aligned}\widehat{V}_{ab}^s := \dot{g}_{ab}g_{ab}^{-1} &= \begin{bmatrix}\dot{R}_{ab} & \dot{p}_{ab}\\ 0 & 0\end{bmatrix}\begin{bmatrix}R_{ab}^T & -R_{ab}^T p_{ab}\\ 0 & 1\end{bmatrix}\\ &= \begin{bmatrix}\dot{R}_{ab}R_{ab}^T & -\dot{R}_{ab}R_{ab}^T p_{ab} + \dot{p}_{ab}\\ 0 & 0\end{bmatrix}\end{aligned}$$

and applying the "vee" operator, we get the twist coordinates

$$V_{ab}^s = \begin{bmatrix}v_{ab}^s\\ \omega_{ab}^s\end{bmatrix} = \begin{bmatrix}-\dot{R}_{ab}R_{ab}^T p_{ab} + \dot{p}_{ab}\\ (\dot{R}_{ab}R_{ab}^T)^\vee\end{bmatrix} \tag{5}$$

2.2 Body Velocity

Say we'd also like to express our velocity with respect to the body frame. Your intuition should tell you that the velocity of the point q with respect to frame B is just 0: since q and B are moving together, they have no velocity with respect to each other. Thus, the velocity of q with respect to frame B is not how we define the body velocity. Instead, we say that the body velocity is the velocity of the point q relative to A, but now expressed in terms of B. As a result, it ends up being a simple coordinate frame transformation of the velocity:

$$v_{q_b}(t) = g_{ab}^{-1}(t) v_{q_a}(t) \tag{6}$$

Again, the notation in the textbook is a bit misleading, with $v_{q_a}(t) \coloneqq \dot{q}_a(t)$, but $v_{q_b}(t) \neq \dot{q}_b = 0$.

We can also find a *body velocity* $\widehat{V}_{ab}^b$ (which is also a twist) such that

$$\boxed{v_{q_b}(t) = g_{ab}^{-1}(t) v_{q_a}(t) = \underbrace{g_{ab}^{-1}(t)\dot{g}_{ab}}_{\coloneqq \widehat{V}_{ab}^b} q_b = \widehat{V}_{ab}^b q_b} \tag{7}$$

Again, dropping the time dependency,

$$\widehat{V}_{ab}^b \coloneqq g_{ab}^{-1}(t)\dot{g}_{ab} = \begin{bmatrix} R_{ab}^T \dot{R}_{ab} & R_{ab}^T \dot{p}_{ab} \\ 0 & 0 \end{bmatrix} \tag{8}$$

and the twist coordinates are

$$V_{ab}^b = \begin{bmatrix} v_{ab}^b \\ \omega_{ab}^b \end{bmatrix} = \begin{bmatrix} R_{ab}^T \dot{p}_{ab} \\ (R_{ab}^T \dot{R}_{ab})^\vee \end{bmatrix} \tag{9}$$

3 Example: One DOF Manipulator

Problem: *Find the spatial and body velocities for the fixed frame A and moving frame B.*

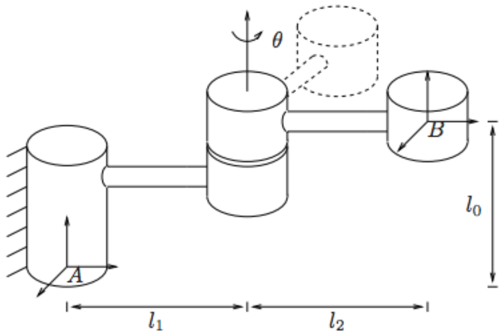


Figure 1: Rigid body motion by rotation about one joint

The following expressions from the forward kinematics might help:

$$g_{ab}(t) = \begin{bmatrix} cos\theta(t) & -sin\theta(t) & 0 & -l_2 sin\theta(t) \\ sin\theta(t) & cos\theta(t) & 0 & l_1 + l_2 cos\theta(t) \\ 0 & 0 & 1 & l_0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$g_{ab}^{-1} = \begin{bmatrix} cos\theta & sin\theta & 0 & -l_1 sin\theta \\ -sin\theta & cos\theta & 0 & -l_2 - l_1 cos\theta \\ 0 & 0 & 1 & -l_0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

From $g_{ab}(t)$, we can calculate the derivative:

$$\dot{g}_{ab} = \begin{bmatrix} -sin\theta & -cos\theta & 0 & -l_2 cos\theta \\ cos\theta & -sin\theta & 0 & -l_2 sin\theta \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \dot{\theta}$$

$$\widehat{V}_{ab}^{s} = \dot{g}_{ab} g_{ab}^{-1} = \begin{bmatrix} 0 & -1 & 0 & l_1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \dot{\theta} \implies V_{ab}^{s} = \begin{bmatrix} l_1 \dot{\theta} \\ 0 \\ 0 \\ 0 \\ 0 \\ \dot{\theta} \end{bmatrix}$$

$$\widehat{V}_{ab}^{b} = g_{ab}^{-1} \dot{g}_{ab} = \begin{bmatrix} 0 & -1 & 0 & -l_2 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \dot{\theta} \implies V_{ab}^{b} = \begin{bmatrix} -l_2 \dot{\theta} \\ 0 \\ 0 \\ 0 \\ 0 \\ \dot{\theta} \end{bmatrix}$$

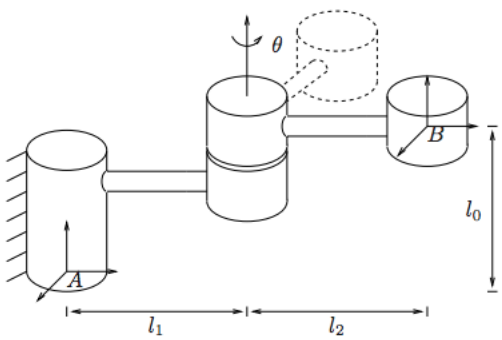
3.1 Interpretation of Twist Coordinates

There's actually a shorter method we can use to find the velocities. It's based on the interpretation of the twist coordinates as summarized in the following table:

Quantity	Interpretation
v^s_{ab}	Velocity of a point at the origin of A if it were attached to the moving frame B
ω^s_{ab}	Angular velocity of B wrt frame A, viewed from A.
v^b_{ab}	Velocity of origin of B wrt frame A, viewed from B.
ω^b_{ab}	Angular velocity of B wrt frame A, viewed from B.

When the velocities are induced by revolute joints, we can imagine circular paths traced out by these joints that help us figure out these values. Let's see how this works with some examples:

Problem: *Find the spatial and body velocity twists for the fixed frame A and moving frame B in Fig. 1 (copied here) using the interpretation of twist coordinates.*



$$V^s_{ab} = \begin{bmatrix} v^s_{ab} \\ \omega^s_{ab} \end{bmatrix}; V^b_{ab} = \begin{bmatrix} v^b_{ab} \\ \omega^b_{ab} \end{bmatrix}$$

v^s_{ab}: A point attached to B that travels through the origin of A has a circular trajectory around the joint axis that passes through the origin of A. Instantaneously at the origin of A, with respect to A and in A-coordinates, the velocity of such a point is $\begin{bmatrix} l_1\dot{\theta} & 0 & 0 \end{bmatrix}^T$.

ω^s_{ab}: With respect to A and expressed in A-coordinates, the angular velocity of B is $\begin{bmatrix} 0 & 0 & \dot{\theta} \end{bmatrix}^T$.

v^b_{ab}: The velocity of the origin of B with respect to A, would be tangential to the circle that the origin of B traces as a function of θ. Thus, as expressed in B-coordinates, this velocity is along the negative x-axis, with coordinates $\begin{bmatrix} -l_2\dot{\theta} & 0 & 0 \end{bmatrix}^T$.

ω^b_{ab}: With respect to A and expressed in B-coordinates, the angular velocity of B is also $\begin{bmatrix} 0 & 0 & \dot{\theta} \end{bmatrix}^T$.

Problem: *Find the spatial and body velocity twists between A to B and also between B to C in Fig. 2 using the interpretation of twist coordinates.*

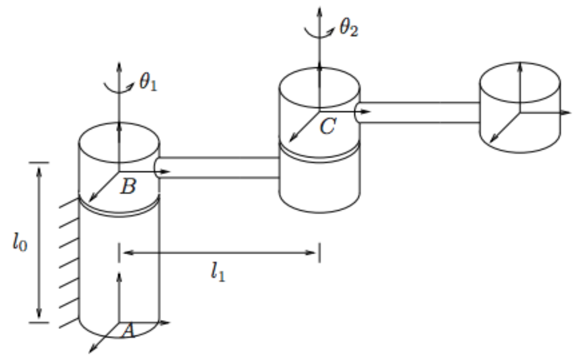


Figure 2: Rigid body motion by rotation about two joints.

$$V^s_{ab} = \begin{bmatrix} v^s_{ab} \\ \omega^s_{ab} \end{bmatrix}; V^b_{ab} = \begin{bmatrix} v^b_{ab} \\ \omega^b_{ab} \end{bmatrix}; V^s_{bc} = \begin{bmatrix} v^s_{bc} \\ \omega^s_{bc} \end{bmatrix}; V^b_{bc} = \begin{bmatrix} v^b_{bc} \\ \omega^b_{bc} \end{bmatrix}$$

v^s_{ab}: A point attached to B that travels through the origin of A will rotate about the axis of joint 1. However, since the origin of A is also on this axis, this point will not move with respect to A. Thus, its velocity is $\begin{bmatrix} 0 & 0 & 0 \end{bmatrix}^T$.

ω^s_{ab}: With respect to A and expressed in A-coordinates, the angular velocity of B is $\begin{bmatrix} 0 & 0 & \dot{\theta}_1 \end{bmatrix}^T$.

v^b_{ab}: The velocity of the origin of B with respect to A would also be zero, $\begin{bmatrix} 0 & 0 & 0 \end{bmatrix}^T$.

ω^b_{ab}: With respect to A and expressed in B-coordinates, the angular velocity of B is also $\begin{bmatrix} 0 & 0 & \dot{\theta}_1 \end{bmatrix}^T$.

v^s_{bc}: A point attached to C that travels through the origin of B will rotate about the axis of joint 2. With respect to frame B, its instantaneous velocity at its origin is $\begin{bmatrix} l_1\dot{\theta} & 0 & 0 \end{bmatrix}^T$.

ω^s_{bc}: With respect to B and expressed in B-coordinates, the angular velocity of C is $\begin{bmatrix} 0 & 0 & \dot{\theta}_2 \end{bmatrix}^T$.

v^b_{bc}: The velocity of the origin of C with respect to B is zero, $\begin{bmatrix} 0 & 0 & 0 \end{bmatrix}^T$.

ω^b_{bc}: With respect to B and expressed in C-coordinates, the angular velocity of C is also $\begin{bmatrix} 0 & 0 & \dot{\theta}_2 \end{bmatrix}^T$.

4 Adjoint Transformations

The Adjoint transformation associated with $g \in SE(3)$ is a 6×6 matrix Ad_g and is defined as follows:

$$Ad_g = \begin{bmatrix} R & \hat{p}R \\ 0 & R \end{bmatrix}$$

The adjoint transformation is invertible and is **used to map twists from one coordinate frame to another**: $\xi' = Ad_g \xi$

Recall that spatial and body velocities are just twists with a coordinate frame transformation applied. As a result, they are related by the adjoint. We have

$$V_{ab}^s = Ad_{g_{ab}} V_{ab}^b$$

Written out, it takes the following form:

$$\boxed{V_{ab}^s = \begin{bmatrix} v_{ab}^s \\ \omega_{ab}^s \end{bmatrix} = \underbrace{\begin{bmatrix} R_{ab} & \widehat{p}_{ab} R_{ab} \\ 0 & R_{ab} \end{bmatrix}}_{:= Ad_{g_{ab}}} \begin{bmatrix} v_{ab}^b \\ \omega_{ab}^b \end{bmatrix} = Ad_{g_{ab}} V_{ab}^b} \tag{10}$$

We can also write out the coordinate frame transformation in terms of the full twist matrix (rather than the twist coordinates):

$$\widehat{V}_{ab}^s = g_{ab} \widehat{V}_{ab}^b g_{ab}^{-1} \tag{11}$$

Therefore, we have

$$Ad_g \xi = (g \cdot \hat{\xi} \cdot g^{-1})^\vee \tag{12}$$

Ad_g is invertible, and its inverse is

$$Ad_{g_{ab}}^{-1} = \begin{bmatrix} R_{ab}^T & -R_{ab}^T \widehat{p} \\ 0 & R_{ab}^T \end{bmatrix} \tag{13}$$

4.1 Coordinate Transformations

We don't compose velocities in the same way as with rigid body transformations—we need to use the adjoints. For spatial velocity composition, we have:

$$V_{ac}^s = V_{ab}^s + Ad_{g_{ab}} V_{bc}^s \tag{14}$$

And with body velocity composition,

$$V_{ac}^b = Ad_{g_{bc}^{-1}} V_{ab}^b + V_{bc}^b \tag{15}$$

Extra practice: *Using your solutions to Problem 3 and the appropriate adjoint transformation, find* V_{ac}^s *in Fig. 2. Your answer should match the solution to Example 2.6 on page 60 in MLS (page 78 in the PDF on the course website).*

Problem: Prove MLS Proposition 2.15: $V_{ac}^b = \mathrm{Ad}_{g_{bc}^{-1}} V_{ab}^b + V_{bc}^b$.

Hint: It may help to take the "hat" of the velocity and use the fact that $g_{ac} = g_{ab} g_{bc}$. *Also,* $(AB)^{-1} = B^{-1}A^{-1}$

We know that $\hat{V}_{ac}^b = g_{ac}^{-1}\dot{g}_{ac}$, and further that $g_{ac} = g_{ab}g_{bc}$. So we can write

$$\begin{aligned}
\hat{V}_{ac}^b &= g_{ac}^{-1}\dot{g}_{ac} \\
&= (g_{ab}g_{bc})^{-1}\frac{d}{dt}(g_{ab}g_{bc}) \\
&= g_{bc}^{-1}g_{ab}^{-1}(\dot{g}_{ab}g_{bc} + g_{ab}\dot{g}_{bc}) \\
&= g_{bc}^{-1}(g_{ab}^{-1}\dot{g}_{ab})g_{bc} + g_{bc}^{-1}\dot{g}_{bc} \\
&= g_{bc}^{-1}\hat{V}_{ab}^b g_{bc} + \hat{V}_{bc}^b
\end{aligned}$$

and now we can "unhat" the above to get the required result

$$V_{ac}^b = \mathrm{Ad}_{g_{bc}^{-1}} V_{ab}^b + V_{bc}^b$$