EE106A Discussion 4: Inverse Kinematics

1 Inverse kinematics

In forward kinematics, we found the expression for $g_{st}(\theta)$ as a function of θ. Now, in inverse kinematics, we are given a desired configuration of the tool frame g_d, and we wish to find the set of θs for which

$$e^{\widehat{\xi_1}\theta_1}...e^{\widehat{\xi_n}\theta_n}g_{st}(0) = g_{st}(\theta) = g_d \tag{1}$$

2 Padan-Kahan subproblems

To solve the inverse kinematics problem, one technique is to distill it into the following three simpler subproblems for which we know the solutions.

2.1 Subproblem 1: Rotation about a single axis

Let ξ be a zero-pitch twist (revolute joint) along ω with unit magnitude, and p, $q \in \mathbb{R}^3$ be two points. If our expression is in the form of

$$e^{\widehat{\xi}\theta}p = q$$

we can uniquely find our θ (1 solution).

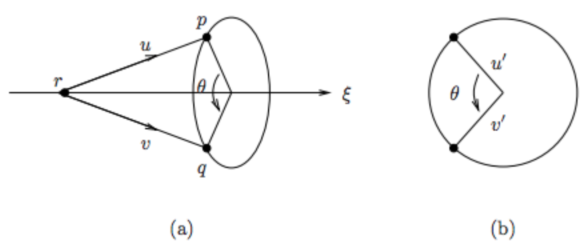

Figure 1: *Subproblem 1: a) Rotate p about the axis of ξ until it is coincident with q. b) Projection of u and v onto the plane perpendicular to the twist axis.*

By the projection formula, $u' = u - \omega\omega^T u$ and $v' = v - \omega\omega^T v$.

By definition of the cross and dot products respectively, $u' \times v' = \omega sin\theta||u'||||v'||$ and $u' \cdot v' = cos\theta||u'||||v'||$. Given that $||\omega|| = 1$, we multiple both sides of the cross product equation by ω^T and divide the two equations to get that $\theta = atan2(\omega^T(u' \times v'), u' \cdot v')$

2.2 Subproblem 2: Rotation about two subsequent axes

Let ξ_1 and ξ_2 be two zero-pitch, unit magnitude twists (revolute joints) with intersecting axes, and p, $q \in \mathbb{R}^3$ be two points. We can find θ_1 and θ_2 if our expression is in the form of

$$e^{\widehat{\xi_1}\theta_1}e^{\widehat{\xi_2}\theta_2}p = q$$

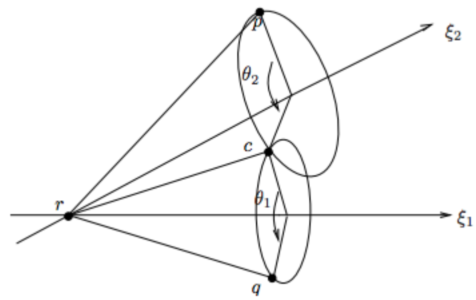

Figure 2: *Subproblem 2: Rotate p around the axis of ξ_2, then around the axis of ξ_1 such that the final location is coincident with q.*

Geometrically, when does there exist zero, one, or multiple solutions to this subproblem?
When the two circles intersect zero, one, or two times respectively.

2.3 Subproblem 3: Rotation to a given distance

Let ξ be a zero-pitch, unit magnitude twist (revolute joint), p, $q \in \mathbb{R}^3$ be two points, and $\delta > 0$. We can find θ if our expression has the following form:

$$||q - e^{\widehat{\xi}\theta}p|| = \delta \text{ or } ||e^{\widehat{\xi}\theta}p - q|| = \delta \tag{2}$$

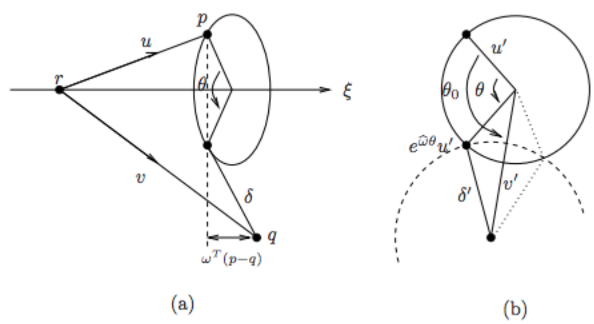

Figure 3: *Subproblem 3: a) Rotate p about the axis of ξ until it is a distance δ from point q. b) Projection onto plane perpendicular to axis.*

• *Geometrically, when does there exist zero, one, or multiple solutions to this subproblem?*
When the circle formed by p's rotation about ξ intersects the sphere of radius δ with center q zero, one, or two times respectively.
To find the solution for θ, we calculate

$$u' = u - \omega\omega^T u$$
$$v' = v - \omega\omega^T v$$

$$\delta = ||v' - e^{\widehat{\omega}\theta}u'||$$

Using the same idea as in subproblem 1, we have that

$$\theta_0 = atan2(\omega^T(u' \times v'), u' \cdot v')$$

By the law of cosines,

$$\delta'^2 = ||u'||^2 + ||v'||^2 - 2||u'||||v'||cos(\theta_0 - \theta)$$

so

$$\theta = \theta_0 \pm cos^{-1}\left(\frac{||u'||^2 + ||v'||^2 - \delta'^2}{2||u'||||v'||}\right)$$

3 Using PK subproblems to solve inverse kinematics

We want to simplify complete inverse kinematics problems into the three subproblems we know how to solve. The full equation becomes more simplified when we apply the kinematics equations to *special points*.

3.1 Trick 1: Apply equations to a point on the axes

If we have a revolute twist ξ and we have a point p on the twist axis, applying the transformation on that point does nothing to it, ie:

$$e^{\widehat{\xi}\theta}p = p \tag{3}$$

For example, if our IK problem is

$$e^{\widehat{\xi_1}\theta_1}e^{\widehat{\xi_2}\theta_2}e^{\widehat{\xi_3}\theta_3} = g \tag{4}$$

then choosing a point p on the axis of ξ_3 yields

$$e^{\widehat{\xi_1}\theta_1}e^{\widehat{\xi_2}\theta_2}p = gp \tag{5}$$

and this is simply Subproblem 2.

3.2 Trick 2: Subtract a point from both sides and take the norm

Remember that rigid motions preserve norm. For example, say we wish to solve the same IK problem as in Eq. 4. If the axes of ξ_1 and ξ_2 intersect at a point q, we can select a point p that is not on the axis of ξ_3 and simplify to the following:

$$\begin{aligned} \delta := ||gp - q|| &= ||e^{\widehat{\xi_1}\theta_1}e^{\widehat{\xi_2}\theta_2}e^{\widehat{\xi_3}\theta_3}p - q|| \\ &= ||e^{\widehat{\xi_1}\theta_1}e^{\widehat{\xi_2}\theta_2}(e^{\widehat{\xi_3}\theta_3}p - q)|| \\ &= ||e^{\widehat{\xi_3}\theta_3}p - q|| \end{aligned} \tag{6}$$

which is just Subproblem 3.

3.3 Trick 3: Prismatic or Screw Joints

It's best to solve these first in general. Use Trick 2 to arrive at the following form:

$$||e^{\widehat{\xi_3}\theta_3}p - q|| = \delta$$

Then, you have $\delta = l_0 + \theta$, where l_0 is the original extension of the arm. You can directly calculate $\theta = \delta - l_0$. If the joint is a screw, you need to be careful about which distance to use for δ (only considering the translation component) and account for your pitch h.

4 SCARA manipulator example

Break down the the inverse kinematics for the SCARA manipulator in Fig. 4 into simpler PK subproblems.

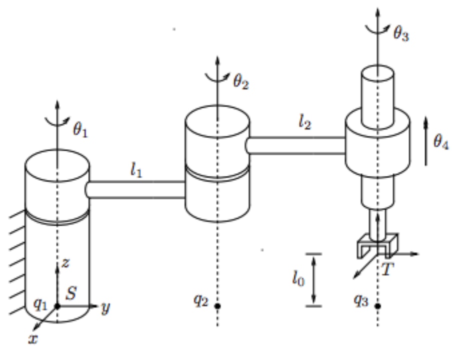

Figure 4: SCARA manipulator.

Step 1: Solve for θ_4
The forward kinematics equation for the SCARA manipulator is

$$g_{st}(\theta) = e^{\hat{\xi}_1\theta_1} \dots e^{\hat{\xi}_4\theta_4} g_{st}(0) = g_d$$

We can visually see that the only joint that affects the z-position of the end effector is ξ_4. As a result, $\theta_4 = z - l_0$, where z comes from $g_{st}(\theta)$.

Step 2: Solve for θ_2
Once θ_4 is known, we can rearrange the FK equation to read

$$e^{\hat{\xi}_1\theta_1} e^{\hat{\xi}_2\theta_2} \dots e^{\hat{\xi}_3\theta_3} = g_d g_{st}^{-1}(0) e^{-\hat{\xi}_4\theta_4} =: g_1$$

Let q_3 be a point on the axis of ξ_3 and q_1 be a point on the axis of ξ_1. (Trick 2) Applying the equation above to q_3, subtracting q_1 from both sides, and applying norms, we get

$$\begin{aligned}
||e^{\hat{\xi}_1\theta_1} e^{\hat{\xi}_2\theta_2} e^{\hat{\xi}_3\theta_3} q_3 - q_1|| &= ||e^{\hat{\xi}_1\theta_1} e^{\hat{\xi}_2\theta_2} q_3 - q_1|| \\
&= ||e^{\hat{\xi}_1\theta_1} (e^{\hat{\xi}_2\theta_2} q_3 - q_1)|| \\
&= ||e^{\hat{\xi}_1\theta_1} (e^{\hat{\xi}_2\theta_2} q_3 - q_1)|| \\
&= ||e^{\hat{\xi}_2\theta_2} q_3 - q_1|| \\
&= ||g_1 q_3 - q_1|| = \delta
\end{aligned}$$

This is exactly in the form of Subproblem 3 and gives us the value of θ_2!

Step 3: Solve for θ_1
We can now find θ_1 by applying the FK equation to a point on the axis of ξ_3:

$$e^{\hat{\xi}_1\theta_1} e^{\hat{\xi}_2\theta_2} e^{\hat{\xi}_3\theta_3} q_3 = e^{\hat{\xi}_1\theta_1} (e^{\hat{\xi}_2\theta_2} q_3) = g_1 q_3$$

This is in the form of Subproblem 1!

Step 4: Solve for θ_3

Finally, we arrange the equations to shift the known θ_1 and θ_2 to the right-hand side:

$$e^{\hat{\xi}_3\theta_3} = e^{-\hat{\xi}_2\theta_2}e^{-\hat{\xi}_1\theta_1}g_d g_{st}^{-1}(\theta)e^{-\hat{\xi}_4\theta_4}$$

We can apply this equation to any point p that's not on the axis of ξ_3 and apply Subproblem 1 to find our answer!

The total number of possible solutions will be $1 \times 2 \times 1 \times 1 = 2$.

5 Elbow manipulator example

Break down the inverse kinematics for the elbow manipulator in Fig. 5 into simpler PK subproblems. Find the reachable and dexterous workspaces.

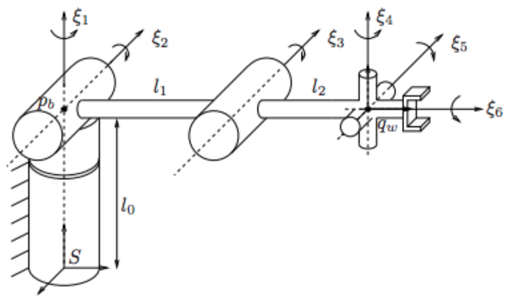

Figure 5: Elbow manipulator.

The dexterous workspace is a hollow sphere with inner radius $l_1 - l_2$ and outer radius $l_1 + l_2$. This is the same as the reachable workspace because our wrist is fully dexterous.

The elbow manipulator in 5 consists of a three degree of freedom manipulator with a spherical wrist. This special structure simplifies the inverse kinematics and fits nicely with the subproblems presented earlier. The equation we wish to solve is

$$gst(\theta) = e^{\hat{\xi}_1\theta_1} \dots e^{\hat{\xi}_6\theta_6} g_{st}(0) = g_d$$

where $g_d \in SE(3)$ is the desired configuration of the tool frame. Postmultiplying this equation by $g_{st}^{-1}(0)$ isolates the exponential maps:

$$e^{\hat{\xi}_1\theta_1} \dots e^{\hat{\xi}_6\theta_6} = g_d g_{st}^{-1}(0) =: g_1$$

We determine the requisite joint angles in four steps:

Step 1: Solve for the elbow angle, θ_3
Apply both sides of the above equation to a point $p_w \in \mathbb{R}^3$, which is the common point of intersection for the wrist axes (trick 1). Since $exp(\hat{\xi}\theta)p_w = p_w$ if pw is on the axis of ξ, this yields

$$e^{\hat{\xi}_1\theta_1} e^{\hat{\xi}_2\theta_2} e^{\hat{\xi}_3\theta_3} p_w = g_1 p_w$$

Subtract from both sides of equation a point p_B, which is at the intersection of the first two axes:

$$e^{\hat{\xi}_1\theta_1} e^{\hat{\xi}_2\theta_2} e^{\hat{\xi}_3\theta_3} p_w - p_b = e^{\hat{\xi}_1\theta_1} e^{\hat{\xi}_2\theta_2} (e^{\hat{\xi}_3\theta_3} p_w - p_b) = g_1 p_w - p_b$$

Using the property that the distance between points is preserved by rigid motions, take the magnitude of both sides of the equation:

$$||g_1 p_w - p_b|| = ||e^{\hat{\xi}_3\theta_3} p_w - p_b||$$

This is in the form of Subproblem 3! We can apply the subproblem and solve for θ_3.

Step 2: Solve for θ_1 and θ_2
Since θ_3 is known, the equation above becomes

$$e^{\hat{\xi}_1\theta_1} e^{\hat{\xi}_2\theta_2} (e^{\hat{\xi}_3\theta_3} p_w) = g_1 p_w$$

We can now apply Subproblem 2! $p = e^{\hat{\xi}_3\theta_3}p_w$ and $q = g_1 p_w$.

Step 3: Solve for 2 of the 3 wrist angles
The remaining kinematics can be written as

$$e^{\hat{\xi}_4\theta_4}e^{\hat{\xi}_5\theta_5}e^{\hat{\xi}_6\theta_6} = e^{-\hat{\xi}_1\theta_1}e^{-\hat{\xi}_2\theta_2}e^{-\hat{\xi}_3\theta_3}g_d g_{st}^{-1}(0) =: g_2$$

Apply both sides of the equation to a point p that is on the axis of ξ_6 but *not* on the ξ_4, ξ_5 axes. This gives

$$e^{\hat{\xi}_4\theta_4}e^{\hat{\xi}_5\theta_5}p = g_2 p$$

We can now just apply Subproblem 2 to find θ_4 and θ_5

Step 4: Solve for the remaining wrist angle
The only remaining unknown is θ_6. Rearranging the kinematics equation and applying both sides to any point p that is not on the axis of ξ_6,

$$e^{\hat{\xi}_6\theta_6} = e^{-\hat{\xi}_5\theta_5}e^{-\hat{\xi}_4\theta_4}\dots e^{-\hat{\xi}_1\theta_1}g_d g_{st}^{-1}(0)p =: q$$

We can just apply Subproblem 1 to find θ_6!

The total number of solutions is $2 \times 2 \times 2 = 8$.