EE106A Discussion 4: Inverse Kinematics

1 Inverse kinematics

In forward kinematics, we found the expression for $g_{st}(\theta)$ as a function of θ. Now, in inverse kinematics, we are given a desired configuration of the tool frame g_d, and we wish to find the set of θs for which

$$e^{\widehat{\xi_1}\theta_1}...e^{\widehat{\xi_n}\theta_n}g_{st}(0) = g_{st}(\theta) = g_d \tag{1}$$

2 Padan-Kahan subproblems

To solve the inverse kinematics problem, one technique is to distill it into the following three simpler subproblems for which we know the solutions.

2.1 Subproblem 1: Rotation about a single axis

Let ξ be a zero-pitch twist (revolute joint) along ω with unit magnitude, and p, $q \in \mathbb{R}^3$ be two points. If our expression is in the form of

$$e^{\widehat{\xi}\theta}p = q$$

we can uniquely find our θ (1 solution).

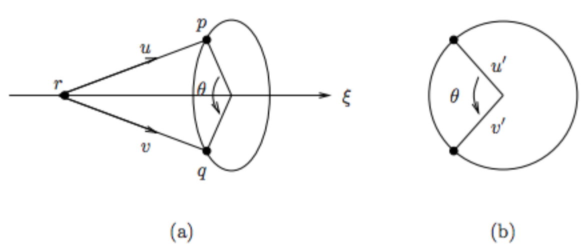

Figure 1: *Subproblem 1: a) Rotate p about the axis of ξ until it is coincident with q. b) Projection of u and v onto the plane perpendicular to the twist axis.*

By the projection formula, $u' = u - \omega\omega^T u$ and $v' = v - \omega\omega^T v$.

By definition of the cross and dot products respectively, $u' \times v' = \omega sin\theta||u'||||v'||$ and $u' \cdot v' = cos\theta||u'||||v'||$. Given that $||\omega|| = 1$, we multiple both sides of the cross product equation by ω^T and divide the two equations to get that $\theta = atan2(\omega^T(u' \times v'), u' \cdot v')$

2.2 Subproblem 2: Rotation about two subsequent axes

Let ξ_1 and ξ_2 be two zero-pitch, unit magnitude twists (revolute joints) with intersecting axes, and p, $q \in \mathbb{R}^3$ be two points. We can find θ_1 and θ_2 if our expression is in the form of

$$e^{\widehat{\xi_1}\theta_1}e^{\widehat{\xi_2}\theta_2}p = q$$

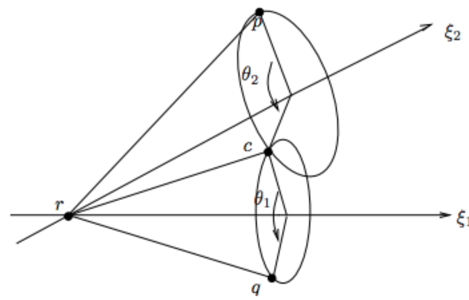

Figure 2: *Subproblem 2: Rotate p around the axis of ξ_2, then around the axis of ξ_1 such that the final location is coincident with q.*

Geometrically, when does there exist zero, one, or multiple solutions to this subproblem?

2.3 Subproblem 3: Rotation to a given distance

Let ξ be a zero-pitch, unit magnitude twist (revolute joint), p, $q \in \mathbb{R}^3$ be two points, and $\delta > 0$. We can find θ if our expression has the following form:

$$||q - e^{\widehat{\xi}\theta} p|| = \delta \;\text{ or }\; ||e^{\widehat{\xi}\theta} p - q|| = \delta \tag{2}$$

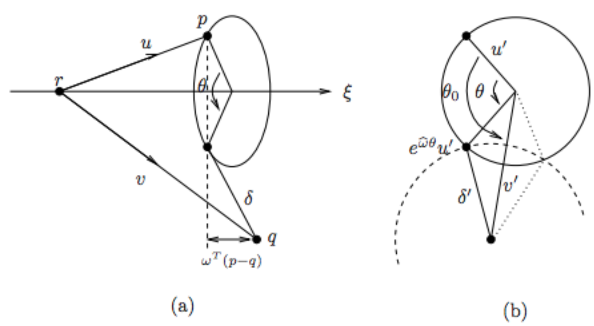

Figure 3: *Subproblem 3: a) Rotate p about the axis of ξ until it is a distance δ from point q. b) Projection onto plane perpendicular to axis.*

- *Geometrically, when does there exist zero, one, or multiple solutions to this subproblem?*

To find the solution for θ, we calculate

$$u' = u - \omega\omega^T u$$
$$v' = v - \omega\omega^T v$$

$$\delta = ||v' - e^{\widehat{\omega}\theta} u'||$$

Using the same idea as in subproblem 1, we have that

$$\theta_0 = atan2(\omega^T(u' \times v'), u' \cdot v')$$

By the law of cosines,

$$\delta'^2 = ||u'||^2 + ||v'||^2 - 2||u'||||v'||cos(\theta_0 - \theta)$$

so

$$\theta = \theta_0 \pm cos^{-1}\left(\frac{||u'||^2 + ||v'||^2 - \delta'^2}{2||u'||||v'||}\right)$$

3 Using PK subproblems to solve inverse kinematics

We want to simplify complete inverse kinematics problems into the three subproblems we know how to solve. The full equation becomes more simplified when we apply the kinematics equations to *special points*.

3.1 Trick 1: Apply equations to a point on the axes

If we have a revolute twist ξ and we have a point p on the twist axis, applying the transformation on that point does nothing to it, ie:

$$e^{\widehat{\xi}\theta}p = p \tag{3}$$

For example, if our IK problem is

$$e^{\widehat{\xi}_1\theta_1}e^{\widehat{\xi}_2\theta_2}e^{\widehat{\xi}_3\theta_3} = g \tag{4}$$

then choosing a point p on the axis of ξ_3 yields

$$e^{\widehat{\xi}_1\theta_1}e^{\widehat{\xi}_2\theta_2}p = gp \tag{5}$$

and this is simply Subproblem 2.

3.2 Trick 2: Subtract a point from both sides and take the norm

Remember that rigid motions preserve norm. For example, say we wish to solve the same IK problem as in Eq. 4. If the axes of ξ_1 and ξ_2 intersect at a point q, we can select a point p that is not on the axis of ξ_3 and simplify to the following:

$$\begin{aligned}\delta := ||gp - q|| &= ||e^{\widehat{\xi}_1\theta_1}e^{\widehat{\xi}_2\theta_2}e^{\widehat{\xi}_3\theta_3}p - q|| \\ &= ||e^{\widehat{\xi}_1\theta_1}e^{\widehat{\xi}_2\theta_2}(e^{\widehat{\xi}_3\theta_3}p - q)|| \\ &= ||e^{\widehat{\xi}_3\theta_3}p - q||\end{aligned} \tag{6}$$

which is just Subproblem 3.

3.3 Trick 3: Prismatic or Screw Joints

It's best to solve these first in general. Use Trick 2 to arrive at the following form:

$$||e^{\widehat{\xi}_3\theta_3}p - q|| = \delta$$

Then, you have $\delta = l_0 + \theta$, where l_0 is the original extension of the arm. You can directly calculate $\theta = \delta - l_0$. If the joint is a screw, you need to be careful about which distance to use for δ (only considering the translation component) and account for your pitch h.

4 SCARA manipulator example

Break down the the inverse kinematics for the SCARA manipulator in Fig. 4 into simpler PK sub-problems.

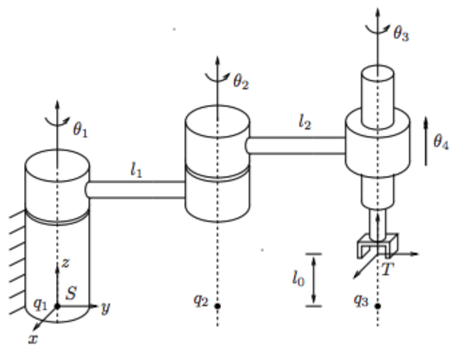

Figure 4: SCARA manipulator.

5 Elbow manipulator example

Break down the inverse kinematics for the elbow manipulator in Fig. 5 into simpler PK subproblems. Find the reachable and dexterous workspaces.

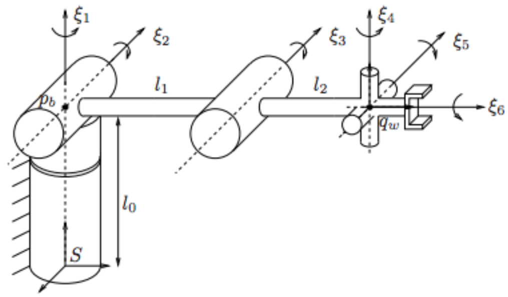

Figure 5: Elbow manipulator.