1 Inverse kinematics

In forward kinematics, we found the expression for \( g_{st}(\theta) \) as a function of \( \theta \). Now, in inverse kinematics, we are given a desired configuration of the tool frame \( g_d \), and we wish to find the set of \( \theta \)s for which

\[
g_{st}(\theta) = g_d
\]  

(1)

Given:
• Desired configuration
  ○ We know where we want our tool to end up
  ○ Ex. In position to grab a box on the table
  
• Also know details about the robot itself
  ○ I.e., we know the twists and starting configuration

\[
\xi_1 \ldots \xi_n \quad \xi_{st}(0)
\]

Desired:
• How do we angle each individual joint to get us there?
  ○ Allow us to move the robot to position it properly
  ○ Find thetas

\[
\theta_1 \ldots \theta_n
\]

2 Padan-Kahan subproblems

To solve the inverse kinematics problem, one technique is to distill it into the following three simpler subproblems for which we know the solutions.

• We know the solutions to some basic inverse kinematics problems
  ○ If our problem is in the form of one of these basic ones, we can find theta

• Can we reduce the super complicated robot problem down to the basic ones?
Subproblems Overview

• Subproblem 1
  ○ Rotate about some fixed axis
  ○ Pure rotation about axis
    \[ \mathbf{e} \cdot \mathbf{p} = q \]
    \[ \theta = \text{atan2}((u' \times v'), u' \cdot v') \]
    (Won’t need to ever compute this)

• Subproblem 2
  ○ Rotate about 2 intersecting axes
    \[ \theta_1, \theta_2 \text{ solutions} \]
    \[ \theta_1, \theta_2 = \text{# See textbook} \]

• Subproblem 3
  ○ Move one point to a specified distance from another
    \[ \| e^{i\theta_1} \mathbf{p} - \mathbf{q} \| = 8 \]
    Distance btwn. rotated \( \mathbf{p} \) & \( \mathbf{q} \) = 8
    \[ \leq 2 \text{ solutions} \]
    \[ \theta = \text{# See worksheet} \]
Okay, so we know we can solve these subproblems. How does that help me with a large robot?

Great question. Our goal is to try to reduce the number of unknowns.

- Use specially chosen points
- Reduce the problem to only 1 or 2 unknown thetas
- Apply subproblems to solve for remaining variables

**Tricks**

\[ e^{\hat{\theta}_1} \ldots e^{\hat{\theta}_n} g_{st}(0) = g \rightarrow \text{FK solution} \]

\[ \text{Rearrange} \rightarrow g_{st}(0) \text{ is known & invertible} \]

\[ e^{\hat{\theta}_1} \ldots e^{\hat{\theta}_n} = g_d g_{st}^{-1}(0) := g \]

**Trick #1: Choose a clever point (eliminate variables from RHS)**

\[ e^{\hat{\theta}_0} p = p \]

\[ \Rightarrow \text{p is on the axis} \]

\[ \text{Rotation won't change location of } p \]

\[ e^{\hat{\theta}_1} e^{\hat{\theta}_2} e^{\hat{\theta}_3} p = g \cdot p \]

\[ g_{z3} = e^{\hat{\theta}_3} \]

\[ g_{z2} = e^{\hat{\theta}_2} \]

\[ g_{z1} = e^{\hat{\theta}_1} \]

\[ \rightarrow \text{Use SP2 to solve } \theta_1, \theta_2 \]

\[ e^{\hat{\theta}_3} q = g \cdot q \rightarrow \theta_3 \text{ w/ SP1} \]
Trick 2: Subtract a point from both sides and take norm (eliminate variables from LHS)

Trick 3: Prismatic Joints

- Good to solve these 1st
- Use geometry of robot

Get to this form w/ trick 2:

\[ \| e^{\hat{e}_1 \theta} p - q \| = \delta \]

\[ \delta = l_1 + \theta \]

\[ \theta = \delta - l_1 \]
4 SCARA manipulator example

Break down the inverse kinematics for the SCARA manipulator in Fig. 4 into simpler PK sub-problems.

**Figure 4: SCARA manipulator.**

\[
\begin{align*}
\hat{e}_{101} &\cdot \hat{e}_{202} \cdot \hat{e}_{303} \cdot e^{404} \cdot g_{d}(0) = g_{d} \\
\end{align*}
\]

**Step 1: Solve for \( \theta_{4} \)**

- \( \theta_{4} \) is the only joint that causes movement in \( z \)-direction
- We know our final \( z \) position
  \[
  g_{d} = \begin{bmatrix} R & t \\ 0 & 1 \end{bmatrix}
  \]
  \[
  z_{f} = t_{z}
  \]
  - Know initial \( z = l_{o} \)
  \[
  \theta_{4} = z_{f} - z_{i} = t_{z} - l_{o}
  \]

\( \checkmark \) solved for \( \theta_{4} \)
Step 2: Solve for $\theta_2$

$$e^{\hat{\Theta}_1} e^{\hat{\Theta}_2} e^{\hat{\Theta}_3} = g_{d_1}^{-1} g_{d_2}^{-1} = \mathbf{g}_2$$

Solve $L_3 = g_1 L_3$

$L_3$ is on $\xi_3$

$$e^{\hat{\Theta}_1} e^{\hat{\Theta}_2} e^{\hat{\Theta}_3} L_3 = g_1 L_3$$

Step 3: Solve for $\Theta_1$

$$e^{\hat{\Theta}_1} e^{\hat{\Theta}_2} e^{\hat{\Theta}_3} L_3 = g_1 L_3$$

$\Rightarrow$ Solve $\Theta_2$ w/ SP 3

Magnitude

$$|| e^{\hat{\Theta}_1} (e^{\hat{\Theta}_2} L_3 - q) || = || g_1 L_3 - q_1 ||$$

Rigid transform

$$|| e^{\hat{\Theta}_2} L_3 - q_1 || = || g_1 L_3 - q_1 ||$$
We know \( \Theta_2 \)

\[
\begin{align*}
\Theta_1, \Theta_2, \Theta_3 \quad &\text{solve for } \Theta_1, w/ \text{SP} \\
\Theta_3 \quad &\text{solve } \Theta_3 \\
\end{align*}
\]

\[
e^{\frac{\Theta_1}{3}} (e^{\frac{\Theta_2}{2}} q_3) = q_1 q_3
\]

\[
\text{Know this value!}
\]

\[
\Rightarrow \text{ Solve for } \Theta_1, w/ \text{SP} \\
\]

\[
\Theta_3 \quad w/ \text{SP} \\
\]

\[
\text{Max # of solns: } 1 \times 2 \times 1 \times 1 = 2
\]

\[
\text{Step 4: Solve } \Theta_3 \\
\]

\[
e^{\frac{\Theta_3}{3}} = e^{-\frac{\Theta_2}{3}} e^{-\frac{\Theta_1}{3}} e^{-1} e^{-\frac{\Theta_4}{3}}
\]

\[
\Rightarrow \quad g_2 = g_2
\]

\[
\Rightarrow \Theta_3 \quad w/ \text{SP} \\
\]

\[
\quad \text{Apply this to any pt. not on } \Xi_3
\]

\[
e^{\frac{\Theta_3}{3}} q_2 = g_2 q_2
\]
5 Elbow manipulator example

Break down the inverse kinematics for the elbow manipulator in Fig. 5 into simpler PK subproblems. Find the reachable and dexterous workspaces.

Figure 5: Elbow manipulator.