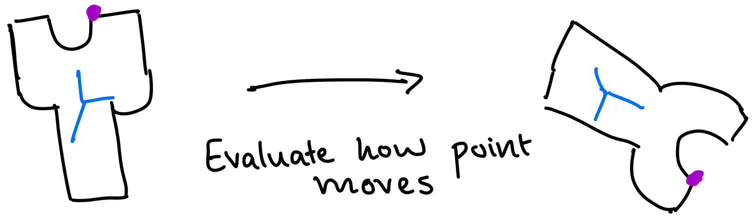
A Summary



- **Rigid body transformations** preserve orientation and direction
- They're affine transformations (Rx + p), rotation then translation
- Points can translate, but vectors simply rotate (since they only represent direction)

- **Homogeneous coordinates** can help us represent this movement

$$p = \begin{bmatrix} p_1 \\ p_2 \\ p_3 \\ 1 \end{bmatrix} \qquad \vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ 0 \end{bmatrix}$$

- Now we can represent rigid transformations for both points and vectors using a single matrix (convert from affine form to linear form)

$$q_a = \begin{bmatrix} q_a \\ 1 \end{bmatrix} = \begin{bmatrix} R_{ab} & p_{ab} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} q_b \\ 1 \end{bmatrix} = g_{ab} q_b$$

$$g = \begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix}$$

- Can stack and invert

$$g_{AC} = g_{AB} g_{BC}$$

$$g_{AC} = g_{CA}^{-1}$$

$$g^{-1} = \begin{bmatrix} R^T & -R^T p \\ 0 & 1 \end{bmatrix}$$

- If we want to parametrize our motion by time, then we can use **exponential coordinates** to generate our transformation matrices

$$g \rightarrow g(t)$$

- Create rotation matrix:

$$R(t) = e^{\hat{\omega} t}$$

ω = axis of rotation

* Same as the Rodrigues Formula

- Can also create homogeneous transformation matrix
- Use the twist (both linear and angular velocity)

$$\xi = \begin{bmatrix} v \\ \omega \end{bmatrix} \in \mathbb{R}^6 \qquad \hat{\xi} = \begin{bmatrix} \hat{\omega} & v \\ 0 & 0 \end{bmatrix}$$

  - Pure rotation (revolute joint)

$$\xi = \begin{bmatrix} -\omega \times q \\ \omega \end{bmatrix}$$

  - Pure translation (prismatic joint)

$$\xi = \begin{bmatrix} v \\ 0 \end{bmatrix}$$

  - Rotation and translation (screw)

$$\xi = \begin{bmatrix} -\omega \times q + h\omega \\ \omega \end{bmatrix}$$

Exponential coordinates:

(ξ, θ)

ξ → How we're rotating

θ ↳ How much we have rotated

$$g = e^{\hat{\xi}\theta}$$

$$p(t) = e^{\hat{\xi} t} p(0)$$

$$e^{\hat{\xi}\theta} = \begin{cases} \begin{bmatrix} I & v\theta \\ 0 & 1 \end{bmatrix} & \omega = 0 \\ \begin{bmatrix} e^{\hat{\omega}\theta} & \left(I - e^{\hat{\omega}\theta}\right)(\omega \times v) + \omega\omega^T v\theta \\ 0 & 1 \end{bmatrix} & \omega \neq 0,\ ||\omega|| = 1 \end{cases}$$

Discussion 2: Exponential Coordinates

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1. Rigid Body Transformations

- **Length-Preserving**
  - **All points stay the same distance** from each other

Transform G

$\forall$ points $p, q \in \mathbb{R}^3$, $\|p - q\| = \|G(p) - G(q)\|$

Distance before & after stays the same

- **Orientation-Preserving**
  - **Points don't switch positions**
  - Same angle relative to each other
  - If your camera is on the top of your phone, it stays on the top

$\forall$ vectors $v, w \in \mathbb{R}^3$, $G(v \times w) = G(v) \times G(w)$

- In other words, a rigid body stays rigid. It's a solid solid.
- **Rotations**, **translations,** and **both** are rigid body transformations

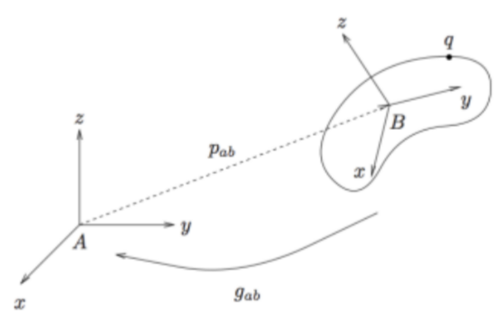
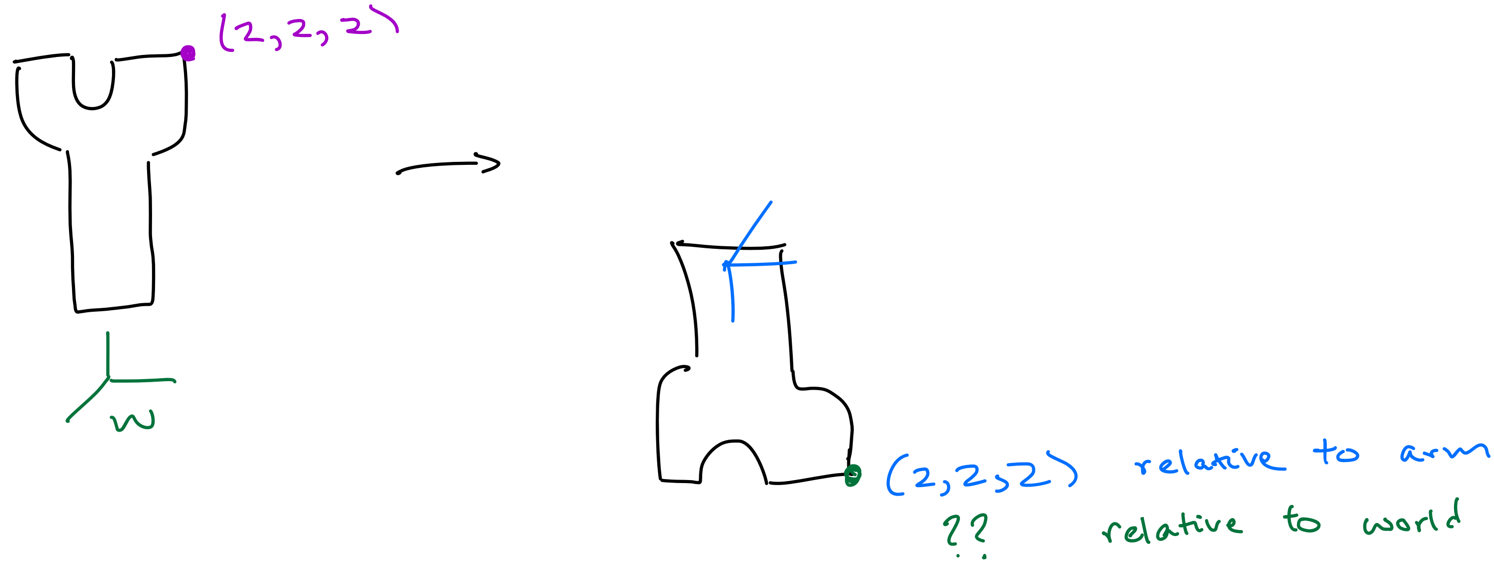


Figure 1: A rigid body transformation.

Rigid Transformation of a Point

- We can move and rotate a coordinate frame
- Points on that frame move and rotate with it

Exercise: *Write out the equation for an affine rigid body transformation of a point. Apply this to a robot arm that has rotated π radians about the y-axis and translated 1 unit in the y-direction. Find the new location of a sensor originally located at $[2, 2, 2]^T$.*

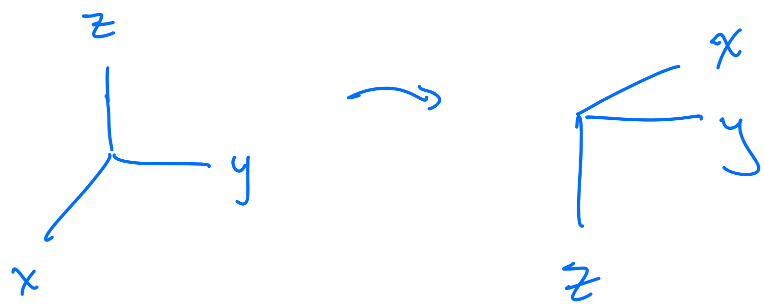


R_{AB}

t_{AB} $\longrightarrow$ $p' = R_{AB} \cdot p + t_{AB}$

p

$R_{AB} = R_y(\pi)$



$$R_{AB} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

↳ new x-axis vector wrt original axes

$t_{AB} \rightarrow$ 1 unit in y-direction

$$= \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \rightarrow \text{wrt original axes}$$

$$p' = R_{AB} \cdot p + t_{AB}$$

$$= \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} -2 \\ 2 \\ -2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$= \boxed{\begin{bmatrix} -2 \\ 3 \\ -2 \end{bmatrix}} \rightarrow$$ New location of sensor with respect to world frame

Affine : $Ax + b$

Rigid Transformation of a Vector

- Vectors only have direction, no positional information

Exercise: *How can we modify the rigid body transformation to apply to vectors?*

$v' = Rv$ (no translation)

$v = s - r$ (subtraction of 2 points)

$$
\begin{aligned}
G(v) &= G(s-r) \\
&= G(s) - G(r) \\
&= R_{AB} \cdot s + t_{AB} - (R_{AB} \cdot r + t_{AB}) \\
&= R_{AB} \cdot s - R_{AB} \cdot r \\
&= R_{AB}(s-r) = \boxed{R_{AB} \cdot v}
\end{aligned}
$$

Homogeneous Coordinates

- Can be used with both points and vectors
  - 4-dimensional array

Affine: $Ax + b$

$\downarrow$

Linear: Ax

$$P_H = \begin{bmatrix} p_x \\ p_y \\ p_z \\ 1 \end{bmatrix} \quad v_H = \begin{bmatrix} v_x \\ v_y \\ v_z \\ 0 \end{bmatrix} \in \mathbb{R}^{4\times 1}$$

Add a 1 for points

0 for vectors

Homogeneous Transformation Matrices

- Combine rotation and translation - homogeneous transformation matrices

$$G_{AB} \in \mathbb{R}^{4\times 4} = \left[\begin{array}{ccc|c} & \underset{\in 3\times 3}{R_{AB}} & & \underset{\in 3\times 1}{t_{AB}} \\ \hline 0 & 0 & 0 & 1 \end{array}\right] = \begin{bmatrix} R & t \\ 0 & 1 \end{bmatrix}$$

Rotation matrix

translation vector

$$G_{AB} \cdot p = \begin{bmatrix} R & t \\ 0 & 1 \end{bmatrix} \begin{bmatrix} p \\ 1 \end{bmatrix} = Rp + t = \begin{bmatrix} p' \\ 1 \end{bmatrix}$$

$$G_{AB} \cdot v = \begin{bmatrix} R & t \\ 0 & 1 \end{bmatrix} \begin{bmatrix} v \\ 0 \end{bmatrix} = Rv + 0 = Rv = \begin{bmatrix} v' \\ 0 \end{bmatrix}$$

- Ex. Flip about y-axis and move 1 unit in y-direction (same as above)

$$R_{AB} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \qquad t_{AB} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

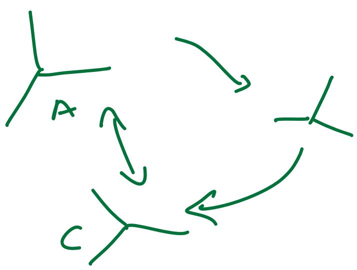
Homogeneous coords of point p

$$p' = G_{AB} \cdot p = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 2 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ 3 \\ -2 \\ 1 \end{bmatrix}$$

Homog. coord for p'

Composition Rule

- Product of 2 rigid body transforms performs both of them
- Go from right to left
- Same as rotation matrices basically, but this also includes translation



$$\boxed{G_{AC} = G_{AB} \cdot G_{BC}}$$

Invertibility

- They're invertible
- Can go from one place to another and back

$$G^{-1} = \begin{bmatrix} R^T & -R^T t \\ 0 & 1 \end{bmatrix}$$

G_{AB}

$\updownarrow$

G_{BA}

$G_{BA} = G_{AB}^{-1}$

2. Exponential Coordinates

Matrix Exponential

- Recall from homework 0 some definitions

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$e^A = \sum_{n=0}^{\infty} \frac{A^n}{n!}$$

$$= I + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \dots$$

$$\frac{dx}{dt} = \dot{x} = Ax \qquad x(0) = x_0$$

$$x(t) = e^{At} x_0$$

Motivation

R

$G = \begin{bmatrix} R & t \\ 0 & 1 \end{bmatrix}$

- We want to construct a **transformation matrix**
- Understand how some point moves with coordinate axes
  - Ex. Where in the world frame does some point on a robot arm end up

→ Make sure it doesn't hit a table!

- But the thing with robots is that they have continuous motion
- A joint can spin around or move forward and back

- **Our transformation matrix changes with movement**
- This means we need the matrix to be a **function of theta** (how much the arm has moved)

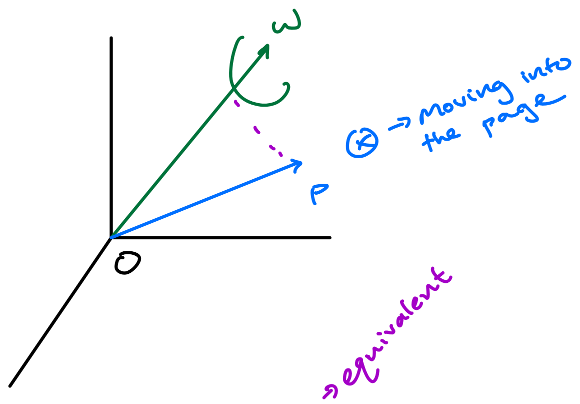
- How do we do that?
- We look at **how the joint moves** (i.e. linear and angular velocities)
- Then integrate!
  - (But this is a DE as we'll see, so it's really an exponential)

Exponential Coordinates for Rotation

- Basically, we're constructing the **rotation matrix** using this technique
- (We'll get to the full homogeneous matrix next)

Problem 1. *Find the rotation matrix $R(\omega, \theta)$ for a rotation about some axis ω by amount θ. How is Rodrigues' formula related?*

☆ Assume unit angular velocity $\|w\| = 1$



* Use θ or θ → equivalent b/c $\|w\| = 1$

Velocity of particle:

$$\dot{p}(t) = \frac{dp}{dt} = \omega \times p(t)$$

→ Cross product gives vector into the page = Linear velocity

$$\frac{dp}{dt} = \hat{\omega}\, p(t)$$

$\hat{\omega}$: skew-symmetric matrix $\in \mathbb{R}^{3\times3}$

$$\boxed{p(t) = e^{\hat{\omega} t}\, p(0)}$$

$e^{\hat{\omega} t}$ → Rotation matrix! parameterized by time

(ω, θ)

↳ Exponential coords for rotation

$\omega \in \mathbb{R}^3$

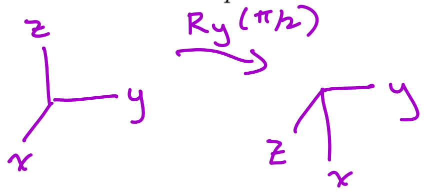
θ → scalar

$$R(\omega, \theta) = e^{\hat{\omega}\theta}$$

= Rodrigues' Formula

$$= I + \hat{\omega}\sin\theta + \hat{\omega}^2(1 - \cos\theta)$$

Exercise: *Find the exponential coordinates* (ω, θ) *of the rotation matrix* $R_y(\frac{\pi}{2})$.



$$\omega = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \qquad \theta = \pi/2 \longrightarrow e^{\hat{\omega}\theta}$$

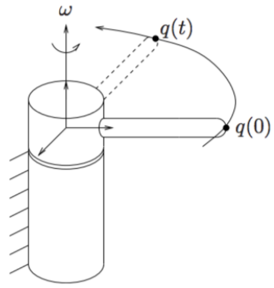


Figure 2: Rotations can happen about any arbitrary axis ω. In this figure the ω axis appears to be coincident with the z-axis, but it can actually be any general vector!

3. Exponential Coordinates for All Rigid Motion

- Usually we want to find more than just the rotation matrix
- See how position changes too
- We want the **full homogeneous transformation**

- We can use **twists** to capture this idea
    - Use **both linear and angular velocities**

Exercise: *Write the expressions for the velocity of the point p (ie. $\dot{p}(t)$) when attached to the revolute joint and attached to the prismatic joint in Fig. 3. Assume that $\omega \in \mathbb{R}^3$, $||\omega|| = 1$, and $q \in \mathbb{R}^3$ is some point along the axis of ω.*

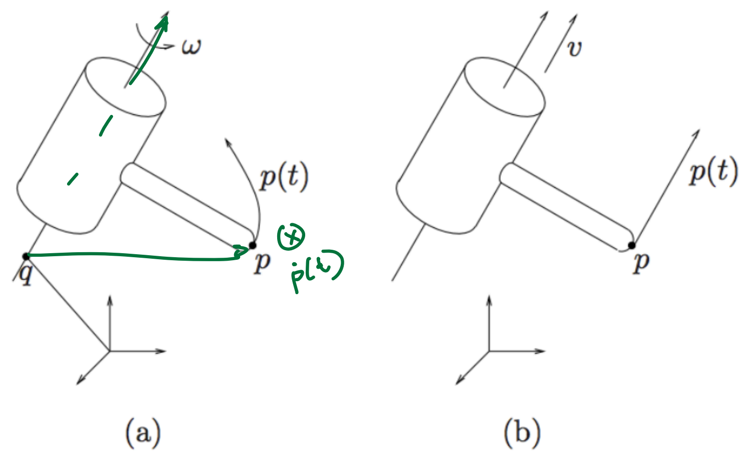


Figure 2: a) A revolute joint and b) a prismatic joint.

Revolute:

$$\dot{p}(t) = \omega \times (p(t) - \underbrace{q}_{\text{any point on } \omega})$$

↳ velocity of p

Prismatic:

$$\dot{p}(t) = v$$

Twist of a Revolute Joint (Rotational Motion)

- Now, let's make the velocity into a DE in homogeneous coordinates

DE:

(put DE from above in matrix-vector form)

$$\underbrace{\begin{bmatrix} \dot{p} \\ 0 \end{bmatrix}}_{\text{velocity vector}} = \underbrace{\begin{bmatrix} \hat{\omega} & -\omega \times q \\ 0 & 0 \end{bmatrix}}_{\hat{\xi}} \underbrace{\begin{bmatrix} p \\ 1 \end{bmatrix}}_{\text{point } p}$$

$$\begin{aligned} \dot{p}(t) &= \omega \times (p(t) - q) \\ &= \omega \times p(t) - \omega \times q \\ &= \hat{\omega} p(t) - \underbrace{\hat{\omega} q}_{3\times 1 \text{ vector}} \end{aligned}$$

Twist of a Prismatic Joint (Linear Motion)

$$\begin{bmatrix} \dot{p} \\ 0 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & v \\ 0 & 0 \end{bmatrix}}_{\hat{\xi}} \begin{bmatrix} p \\ 1 \end{bmatrix}$$

$\dot{p}(t) = v$

$\hat{\xi} \longrightarrow$ Twist

More on Twists

Wedge:
"Hat"

$$\begin{bmatrix} v \\ w \end{bmatrix}^{\wedge} = \begin{bmatrix} \hat{\omega} & v \\ 0 & 0 \end{bmatrix} = \hat{\xi}$$

All info from twist matrices can be captured in a 6x1 vector

Vee:

$$\begin{bmatrix} \hat{\omega} & v \\ 0 & 0 \end{bmatrix}^{\vee} = \begin{bmatrix} v \\ w \end{bmatrix} = \xi \in \mathbb{R}^{6\times 1}$$

$\begin{bmatrix} v \\ w \end{bmatrix}$ $\begin{matrix} \rightarrow \text{Linear vel.} \\ \rightarrow \text{Angular vel.} \end{matrix}$

Exercise: *Find the twist coordinates for a revolute and prismatic joint.*

Revolute:

$$\xi = \begin{bmatrix} v \\ \omega \end{bmatrix} = \begin{bmatrix} -\omega \times q \\ \omega \end{bmatrix} \rightarrow \text{Twist coords}$$

Prismatic:

$$\xi = \begin{bmatrix} v \\ \omega \end{bmatrix} = \begin{bmatrix} v \\ 0 \end{bmatrix} \rightarrow \text{No angular velocity}$$

3.4 Solution to differential equation gives us the exponential map

Problem 5. *Write the general solution to the differential equation* $\dot{\bar{p}} = \widehat{\xi}\bar{p}$*. Then, make use of the fact that* $||\omega|| = 1$ *to reparameterize* t *to be* θ*. Specifically, find the expression for* $p(\theta)$ *in terms of* $p(0)$*.*

$\dot{p} = \hat{\xi} p$ (in homogeneous coords)

$p(t) = e^{\hat{\xi} t} p(0)$

$e^{\hat{\xi} t}$ → Homogeneous Transform Matrix

- It's a mapping of points from initial coordinates to final coordinates after motion with parameter
- Not a mapping between coordinate frames

$$\text{Exp. coords} = (\underbrace{\xi}_{\in 6+1}, \underbrace{\theta}_{\text{scalar}})$$

$$= \left(\begin{bmatrix} v \\ \omega \end{bmatrix}, \theta \right)$$

$$e^{\hat{\xi}\theta} = \begin{cases} \begin{bmatrix} I & v\theta \\ 0 & 1 \end{bmatrix} & \omega = 0 \\ \\ \begin{bmatrix} e^{\hat{\omega}\theta} & \left(I - e^{\hat{\omega}\theta}\right)(\omega \times v) + \omega\omega^T v\theta \\ 0 & 1 \end{bmatrix} & \omega \neq 0, \ ||\omega|| = 1 \end{cases}$$

Screw Motion

→ Chasles' Theorem

- Any rigid body translation can be simplified
- Instead of having a rotation and then a translation
- Finite rotation about some axis and then translation about that axis
  - Axis *l*
  - Magnitude M (like theta)
  - Pitch *h* = ratio of translation : rotation
    - h = 0: pure rotation (revolute joint)
    - h infinite: pure translation (prismatic joint)
- Rotation by M (theta)

The transformation g corresponding to S has the following effect on a point p:

$$gp = q + e^{\hat{\omega}\theta}(p - q) + h\theta\omega \tag{11}$$

Exp. coords. (ξ, θ)

- Translation by hM (apply ratio)

$$g \begin{bmatrix} p \\ 1 \end{bmatrix} = \begin{bmatrix} e^{\hat{\omega}\theta} & (I - e^{\hat{\omega}\theta})q + h\omega\theta \\ 0 & 1 \end{bmatrix} \begin{bmatrix} p \\ 1 \end{bmatrix}$$

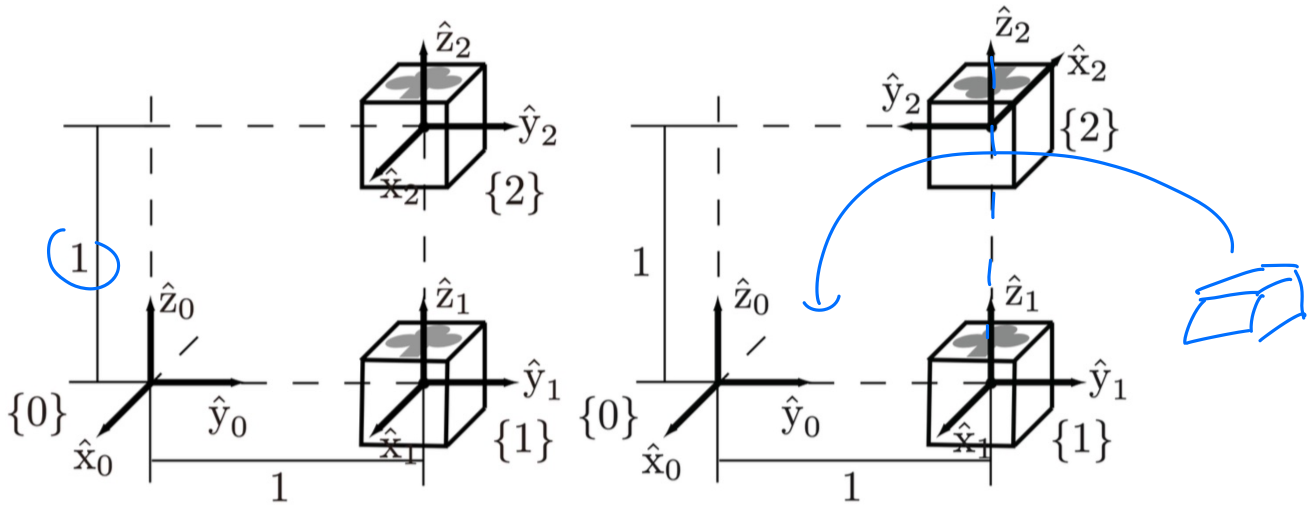
→ Very similar in form to equation above

Every twist ⟺ Equivalent screw

$$\xi = \begin{bmatrix} v \\ \omega \end{bmatrix} = \begin{bmatrix} -\omega \times q + h\omega \\ \omega \end{bmatrix}$$

4 Finding Exponential Coordinates

Figure (4) shows a cube undergoing two different rigid body transformations from frame {1} to frame {2}. In both cases, find a set of exponential coordinates for the rigid body transform that maps the cube from its initial to its final configuration, as viewed from frame {0}. Do this by first finding the equivalent screw motion.



(a) A first screw motion. (b) A second screw motion.

A cube undergoing two screw motions.

a) - Simply a translation up along z-axis

- Translated by 1 unit

- $\xi = \begin{bmatrix} v \\ \omega \end{bmatrix}$

$v \rightarrow$ direction of motion $= \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

$\omega \rightarrow$ angular velocity $= 0$

$\theta = 1$ (moved by 1 unit)

$$(\xi, \theta) = \left(\begin{bmatrix} v \\ \omega \end{bmatrix}, 1 \right) = \left(\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, 1 \right)$$

b) - Translation along z-axis by 1 unit
- Rotation about z by π rad

$$\text{Axis} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$ → Rotation & translation about the same axis (screw motion)

$$\omega = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \qquad \theta = \pi \qquad h = \frac{1}{\pi}$$

↳ Pitch

Ratio trans : rot

$$\xi = \begin{bmatrix} v \\ \omega \end{bmatrix} = \begin{bmatrix} -\omega \times q + h\omega \\ \omega \end{bmatrix}$$

q → any point on axis of rotation

→ most convenient pt. $= \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

$$v = -\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \times \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + \frac{1}{\pi} \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1/\pi \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1/\pi \end{bmatrix}$$

$$(\xi, \theta) = \left(\begin{bmatrix} v \\ \omega \end{bmatrix}, \theta \right)$$

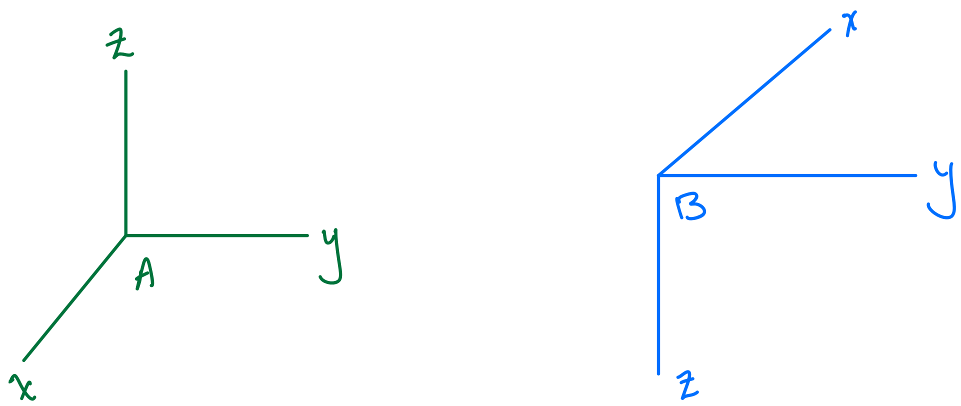
$$= \left(\begin{bmatrix} 1 \\ 0 \\ 1/\pi \\ 0 \\ 0 \\ 1 \end{bmatrix}, \pi \right)$$

$\downarrow$

Plug into $e^{\hat{\xi}\theta}$ to get homog. transform matrix

Equivalent Interpretations of Rotation Matrices

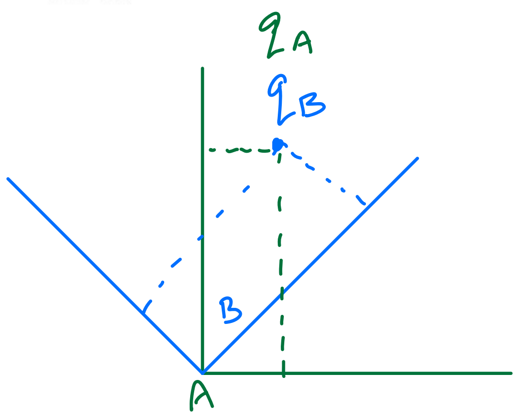
R_{AB} is composed of 3 unit column vectors that represent the B frame in terms of the A frame (x_{AB}, y_{AB}, z_{AB}).



$$R_{AB} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

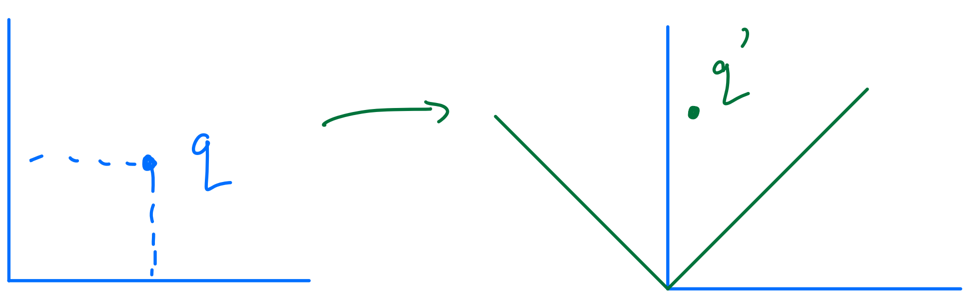
x-axis of B frame in terms of A frame

R_{AB} applied on some point q in the B frame will tell us what that point would be in the A frame: $q_A = R_{AB} q_B$



$q_A = R_{AB} q_B$

If we originally had some point in the standard coordinate frame and then we rotated it, we can find the new location of that point using the rotation matrix: $q' = R_{AB} q$



$q' = R_{AB} q$