A Summary

- **Rigid body transformations** preserve orientation and direction
- They’re affine transformations \((R_x + p)\), rotation then translation
- Points can translate, but vectors simply rotate (since they only represent direction)

- **Homogeneous coordinates** can help us represent this movement

\[
P = \begin{bmatrix} P_1 \\ P_2 \\ P_3 \\ 1 \end{bmatrix} \quad \overrightarrow{V} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ 0 \end{bmatrix}
\]

- Now we can represent rigid transformations for both points and vectors using a single matrix (convert from affine form to linear form)

\[
q_a = \begin{bmatrix} q_a \\ 1 \end{bmatrix} = \begin{bmatrix} R_{ab} & P_{ab} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} q_b \\ 1 \end{bmatrix} = g_{ab} q_b
\]

\[
g = \begin{bmatrix} R & P \\ 0 & 1 \end{bmatrix}
\]

- Can stack and invert

\[
g_{ac} = g_{ab} g_{bc}
\]

\[
g_{ac}^{-1} = g_{ca}
\]

\[
g^{-1} = \begin{bmatrix} R^T & -R^T P \\ 0 & 1 \end{bmatrix}
\]
• If we want to parametrize our motion by time, then we can use **exponential coordinates** to generate our transformation matrices

\[ g \rightarrow g(t) \]

• Create rotation matrix:

\[ R(t) = e^{\hat{w}t} \]

\[ w = \text{axis of rotation} \]

*Same as the Rodrigues Formula*

• Can also create homogeneous transformation matrix

• Use the twist (both linear and angular velocity)

\[ \xi(t) = \begin{bmatrix} v \\ w \end{bmatrix} e^{\hat{\xi}t} \]

\[ \hat{\xi} = \begin{bmatrix} 0 & -w & v \\ w & 0 & 0 \end{bmatrix} \]

○ Pure rotation (revolute joint)

\[ \xi = \begin{bmatrix} -w \times q \\ w \end{bmatrix} \]

○ Pure translation (prismatic joint)

\[ \xi = \begin{bmatrix} v \\ 0 \end{bmatrix} \]

○ Rotation and translation (screw)

\[ \xi = \begin{bmatrix} -w \times q + hw \\ w \end{bmatrix} \]
Discussion 2: Exponential Coordinates

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1. Rigid Body Transformations

• Length-Preserving
  o All points stay the same distance from each other

\[ \forall \text{ points } p, q \in \mathbb{R}^3, \quad \|p - q\| = \|G(p) - G(q)\| \]

• Orientation-Preserving
  o Points don’t switch positions
  o Same angle relative to each other
  o If your camera is on the top of your phone, it stays on the top

\[ \forall \text{ vectors } u, w \in \mathbb{R}^3, \quad G(u \times w) = G(u) \times G(w) \]

• In other words, a rigid body stays rigid. It’s a solid solid.
• Rotations, translations, and both are rigid body transformations

Figure 1: A rigid body transformation.
Rigid Transformation of a Point

- We can move and rotate a coordinate frame
- Points on that frame move and rotate with it

**Exercise:** Write out the equation for an affine rigid body transformation of a point. Apply this to a robot arm that has rotated π radians about the y-axis and translated 1 unit in the y-direction. Find the new location of a sensor originally located at \([2,2,2]^T\).

\[
\begin{align*}
R_{AB} &= R_y(\pi) \\
T_{AB} &\rightarrow \mathbf{p}' = R_{AB} \cdot \mathbf{p} + T_{AB}
\end{align*}
\]
\( t_{AB} \rightarrow \) unit in y-direction
\[
= \begin{bmatrix}
0 \\
1 \\
0 \\
\end{bmatrix} \quad \text{wrt original axes}
\]

\[ P' = R_{AB} \cdot P + t_{AB} \]
\[
= \begin{bmatrix}
-1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & -1 \\
\end{bmatrix} \begin{bmatrix}
2 \\
2 \\
2 \\
\end{bmatrix} + \begin{bmatrix}
0 \\
0 \\
0 \\
\end{bmatrix}
\]
\[
= \begin{bmatrix}
-2 \\
2 \\
-2 \\
\end{bmatrix} + \begin{bmatrix}
0 \\
0 \\
0 \\
\end{bmatrix}
\]
\[
= \begin{bmatrix}
-2 \\
3 \\
-2 \\
\end{bmatrix} \quad \rightarrow \text{New location of sensor with respect to world frame}
\]

Affine : \( Ax + b \)
Rigid Transformation of a Vector

- Vectors only have direction, no positional information

Exercise: How can we modify the rigid body transformation to apply to vectors?

\[ V' = R \cdot V \quad \text{(no translation)} \]

\[ V = s - r \quad \text{(subtraction of 2 points)} \]

\[ G(V) = G(s) - G(r) \]

\[ = R_{AB} \cdot s + t_{AB} - (R_{AB} \cdot r + t_{AB}) \]

\[ = R_{AB} \cdot s - R_{AB} \cdot r \]

\[ = R_{AB}(s - r) = R_{AB} \cdot V \]

Homogeneous Coordinates

- Can be used with both points and vectors
  - 4-dimensional array

Affine: \[ A \cdot x + b \]
Linear: \[ A \cdot x \]

\[ P_H = \begin{bmatrix} P_x \\ P_y \\ P_z \\ 1 \end{bmatrix} \quad V_H = \begin{bmatrix} V_x \\ V_y \\ V_z \\ 0 \end{bmatrix} \quad G \in \mathbb{R}^{4 \times 1} \]

Homogeneous Transformation Matrices

- Combine rotation and translation - homogeneous transformation matrices

\[ G_{AB} \in \mathbb{R}^{4 \times 4} = \begin{bmatrix} R_{AB} & t_{AB} \\ 0 & 1 \end{bmatrix} \]

\[ G_{AB} \cdot P = \begin{bmatrix} R & t \\ 0 & 1 \end{bmatrix} \begin{bmatrix} P \\ 1 \end{bmatrix} = R \cdot P + t = \begin{bmatrix} P' \\ 1 \end{bmatrix} \]

\[ G_{AB} \cdot V = \begin{bmatrix} R & t \\ 0 & 1 \end{bmatrix} \begin{bmatrix} V \\ 0 \end{bmatrix} = R \cdot V + 0 = R \cdot V \]
• Ex. Flip about y-axis and move 1 unit in y-direction (same as above)

\[
R_{AB} = \begin{bmatrix}
-1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & -1
\end{bmatrix} \quad t_{AB} = \begin{bmatrix}
0 \\
1 \\
0
\end{bmatrix}
\]

\[p' = G_{AB} \cdot p = \begin{bmatrix}
-1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & -1
\end{bmatrix} \begin{bmatrix}
2 \\
-2 \\
1
\end{bmatrix} = \begin{bmatrix}
-2 \\
3 \\
-2
\end{bmatrix}
\]

**Composition Rule**

• Product of 2 rigid body transforms performs both of them
• Go from right to left
• Same as rotation matrices basically, but this also includes translation

\[G_{AC} = G_{AB} \cdot G_{BC}\]

**Invertibility**

• They’re invertible
• Can go from one place to another and back

\[G^{-1} = \begin{bmatrix}
R^T & -R^Tt \\
0 & 1
\end{bmatrix}\]

\[G_{BA} = G_{AB}^{-1}\]
2. Exponential Coordinates

Matrix Exponential

- Recall from homework 0 some definitions

\[ e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \]

\[ e^A = \sum_{n=0}^{\infty} \frac{A^n}{n!} \]

\[ = I + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \ldots \]

\[ \frac{dx}{dt} = \dot{x} = Ax \quad x(0) = x_0 \]

\[ x(t) = e^{At} x_0 \]
Motivation

- We want to construct a **transformation matrix**
- Understand how some point moves with coordinate axes
  - Ex. Where in the world frame does some point on a robot arm end up
  - Make sure it doesn’t hit a table!
- But the thing with robots is that they have continuous motion
- A joint can spin around or move forward and back

- **Our transformation matrix changes with movement**
- This means we need the matrix to be a **function of theta** (how much the arm has moved)

- How do we do that?
- We look at **how the joint moves** (i.e. linear and angular velocities)
- Then integrate!
  - (But this is a DE as we’ll see, so it’s really an exponential)
Exponential Coordinates for Rotation

• Basically, we’re constructing the rotation matrix using this technique
• (We’ll get to the full homogeneous matrix next)

Problem 1. Find the rotation matrix $R(\omega, \theta)$ for a rotation about some axis $\omega$ by amount $\theta$. How is Rodrigues’ formula related?

* Assume unit angular velocity

**Velocity of particle:**

\[ \dot{p}(t) = \frac{dp}{dt} = \omega \times p(t) = \text{linear velocity} \]

\[ \frac{dp}{dt} = \hat{\omega} p(t) \]

**skew-symmetric matrix $\in \mathbb{R}^{3 \times 3}$**

\[ p(t) = e^{\hat{\omega} t} p(0) \]

\[ e^{\hat{\omega} t} \rightarrow \text{Rotation matrix parameterized by time} \]

\[ (\omega, \theta) \]

\( \Rightarrow \) Exponential coords for rotation

\[ \omega \in \mathbb{R}^3 \]

\[ \theta \rightarrow \text{scalar} \]

\[ R(\omega, \theta) = e^{\hat{\omega} \theta} = \text{Rodrigues' Formula} = I + \hat{\omega} \sin \theta + \hat{\omega}^2 (1 - \cos \theta) \]
Exercise: Find the exponential coordinates $(\omega, \theta)$ of the rotation matrix $R_y(\pi/2)$.

\[
\begin{bmatrix}
0 \\
1 \\
0
\end{bmatrix}
\]

\[\theta = \pi/2 \rightarrow e^{\hat{\omega} \theta}
\]

Figure 2: Rotations can happen about any arbitrary axis $\omega$. In this figure the $\omega$ axis appears to be coincident with the $z$-axis, but it can actually be any general vector!
3. Exponential Coordinates for All Rigid Motion

- Usually we want to find more than just the rotation matrix
- See how position changes too
- We want the **full homogeneous transformation**

- We can use **twists** to capture this idea
  - Use both linear and angular velocities

**Exercise:** Write the expressions for the velocity of the point \( p \) (i.e. \( \dot{p}(t) \)) when attached to the revolute joint and attached to the prismatic joint in Fig. 3. Assume that \( \omega \in \mathbb{R}^3 \), \( ||\omega|| = 1 \), and \( q \in \mathbb{R}^3 \) is some point along the axis of \( \omega \).

![Image of revolute and prismatic joints]

Figure 2: a) A revolute joint and b) a prismatic joint.

**Twist of a Revolute Joint (Rotational Motion)**

- Now, let’s make the velocity into a DE in homogeneous coordinates

\[
\begin{align*}
\dot{p}(t) &= \omega \times (p(t) - q) \\
\dot{q}(t) &= \omega \
\end{align*}
\]
Twist of a Prismatic Joint (Linear Motion)

\[
\begin{bmatrix}
\dot{p} \\
0
\end{bmatrix} = \begin{bmatrix}
0 & v \\
0 & 0
\end{bmatrix} \begin{bmatrix}
p \\
1
\end{bmatrix}
\]

\[\hat{\xi} \rightarrow \text{Twist}\]

More on Twists

Wedge:

\[
\left[\begin{array}{c}
v \\
w
\end{array}\right]^\wedge = \left[\begin{array}{c}
\hat{\omega} \\
0 \\
0
\end{array}\right] = \hat{\xi}
\]

“Hat”:

\[
\left[\begin{array}{c}
v \\
w
\end{array}\right] = \left[\begin{array}{c}
\hat{\omega} \\
0 \\
0
\end{array}\right] = \hat{\xi}
\]

Vee:

\[
\left[\begin{array}{c}
\hat{\omega} \\
0 \\
0
\end{array}\right]^\vee = \left[\begin{array}{c}
v \\
w
\end{array}\right] = \bar{\xi}
\]

All info from twist matrices can be captured in a 6x1 vector:

\[
\left[\begin{array}{c}
v \\
w
\end{array}\right] \rightarrow \text{Linear vel.}
\]

\[
\left[\begin{array}{c}
\omega
\end{array}\right] \rightarrow \text{Angular vel.}
\]

Exercise: Find the twist coordinates for a revolute and prismatic joint.

Revolute:

\[
\bar{\xi} = \left[\begin{array}{c}
v \\
w
\end{array}\right] = \left[\begin{array}{c}
-\omega \times q \\
w
\end{array}\right] \rightarrow \text{Twist coords}
\]

Prismatic:

\[
\bar{\xi} = \left[\begin{array}{c}
v \\
w
\end{array}\right] = \left[\begin{array}{c}
v \\
0
\end{array}\right] \rightarrow \text{No angular velocity}
\]
3.4 Solution to differential equation gives us the exponential map

**Problem 5.** Write the general solution to the differential equation $\dot{p} = \xi \dot{p}$. Then, make use of the fact that $||\omega|| = 1$ to reparameterize $t$ to be $\theta$. Specifically, find the expression for $p(\theta)$ in terms of $p(0)$.

$$\dot{p} = \xi \dot{p} \quad \text{(in homogeneous coords)}$$

$$p(t) = e^{\xi t} p(0)$$

- It’s a mapping of points from initial coordinates to final coordinates after motion with parameter
- Not a mapping between coordinate frames

$$\text{Exp. coords } = \left( \xi^6, \Theta \right)$$

$$= \left( \begin{bmatrix} \omega \\ \omega \end{bmatrix}, \Theta \right)$$

$$e^{\xi \theta} = \begin{cases} 
\begin{bmatrix} 1 & \nu \theta \\ 0 & 1 \end{bmatrix} & \omega = 0 \\
\begin{bmatrix} e^{\omega \theta} & (I - e^{\omega \theta})(\omega \times \nu) + \omega \nu^T \nu \theta \\ 0 & 1 \end{bmatrix} & \omega \neq 0, \ ||\omega|| = 1 
\end{cases}$$
Screw Motion

- Any rigid body translation can be simplified
- Instead of having a rotation and then a translation
- Finite rotation about some axis and then translation about that axis
  - Axis \( l \)
  - Magnitude \( M \) (like theta)
  - Pitch \( h = \) ratio of translation : rotation
    - \( h = 0: \) pure rotation (revolute joint)
    - \( h \) infinite: pure translation (prismatic joint)
- Rotation by \( M \) (theta)

The transformation \( g \) corresponding to \( S \) has the following effect on a point \( p \):

\[
gp = q + e^{\hat{\omega} \theta} (p - q) + h \theta \omega
\]

- Translation by \( hM \) (apply ratio)

\[
g \begin{bmatrix} p \\ 1 \end{bmatrix} = \begin{bmatrix} e^{\hat{\omega} \theta} & (I - e^{\hat{\omega} \theta})q + hw \theta \\ 0 & 1 \end{bmatrix} \begin{bmatrix} p \\ 1 \end{bmatrix}
\]

\[\Rightarrow \text{very similar in form to equation above}\]

\[\text{Every twist} \quad \leftrightarrow \quad \text{Equivalent screw}\]

\[
\begin{bmatrix} \zeta \\ w \end{bmatrix} = \begin{bmatrix} -w \times q + hw \\ w \end{bmatrix}
\]
4 Finding Exponential Coordinates

Figure (4) shows a cube undergoing two different rigid body transformations from frame \( \{1\} \) to frame \( \{2\} \). In both cases, find a set of exponential coordinates for the rigid body transform that maps the cube from its initial to its final configuration, as viewed from frame \( \{0\} \). Do this by first finding the equivalent screw motion.

(a) A first screw motion. (b) A second screw motion.

A cube undergoing two screw motions.

\( a) \) Simply a translation up along \( z \)-axis
- Translated by 1 unit
- \( \xi = \begin{bmatrix} v \\ w \end{bmatrix} \)

\( v \rightarrow \) direction of motion = \( \begin{bmatrix} 0 \\ 1 \end{bmatrix} \)

\( w \rightarrow \) angular velocity = 0

\( \Theta = 1 \) (moved by 1 unit)

\( (\xi, \Theta) = \left( \begin{bmatrix} v \\ w \end{bmatrix}, 1 \right) = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \)
b) Translation along Z-axis by 1 unit
- Rotation about Z by \( \pi \) rad

\[
\text{Axis} = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \rightarrow \text{Rotation & translation about the same axis (Screw motion)}
\]

\[
\begin{align*}
W &= \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \\
\Theta &= \pi \\
h &= \frac{1}{\pi}
\end{align*}
\]

\[
\begin{bmatrix} v \\ w \end{bmatrix} = \begin{bmatrix} -w \times q + hw \\ w \end{bmatrix}
\]

\( q \rightarrow \) any point on axis of rotation

\( \rightarrow \) most convenient pt. \( q = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \)

\[
V = -\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \times \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + \frac{1}{\pi} \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}
\]

\[
= \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1/\pi \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1/\pi \\ 0 \end{bmatrix}
\]
\[
(x, \theta) = \begin{pmatrix}
\begin{bmatrix} u \\ w \end{bmatrix}, \\
\theta
\end{pmatrix} = \begin{pmatrix}
\begin{bmatrix} 1 \\ 0 \\ \frac{1}{\pi} \\ 0 \end{bmatrix}, \\
\frac{1}{\pi}
\end{pmatrix}
\]

Plug into \( e^{x} \) to get homog. transform matrix
Equivalent Interpretations of Rotation Matrices

\( R_{AB} \) is composed of 3 unit column vectors that represent the \( B \) frame in terms of the \( A \) frame \((x_{AB}, y_{AB}, z_{AB})\).

\[
R_{AB} = \begin{bmatrix}
0 & 0 & 1 \\
0 & 1 & 0 \\
-1 & 0 & 0 \\
\end{bmatrix}
\]

\( R_{AB} \) applied on some point \( q \) in the \( B \) frame will tell us what that point would be in the \( A \) frame: \( q_A = R_{AB}q_B \)

If we originally had some point in the standard coordinate frame and then we rotated it, we can find the new location of that point using the rotation matrix: \( q' = R_{AB}q \)