EE106A Discussion 1: Rotations

1 Frame-specific representations

Points and vectors are described by coordinates that are only meaningful with respect to a corresponding coordinate frame.

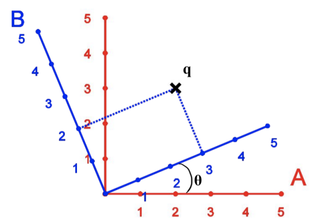


Figure 1: Two coordinate frames A and B

Problem 1. *Write the representation of point q with respect to the coordinate frames A and B, which we denote q_a and q_b respectively.*

$$q_a = [2 \quad 3]^T, q_b = [3 \quad 2]^T$$

2 Rotation matrices

Let's first think solely about the mathematical definition of a rotation matrix before discussing how they are used in practice. A rotation matrix is a matrix that is defined according to two coordinate frames.

Definition 1. Say we have coordinate frame A, defined by its principal axes $\{\boldsymbol{x_a}, \boldsymbol{y_a}, \boldsymbol{z_a}\}$, and frame B, with principal axes $\{\boldsymbol{x_b}, \boldsymbol{y_b}, \boldsymbol{z_b}\}$. Then, we define a rotation matrix R_{ab} to be

$$R_{ab} := [\boldsymbol{x_{ab}} \quad \boldsymbol{y_{ab}} \quad \boldsymbol{z_{ab}}]$$

where $\{\boldsymbol{x_{ab}}, \boldsymbol{y_{ab}}, \boldsymbol{z_{ab}}\}$ are orthonormal principal axes of frame B expressed in the coordinates of frame A.

Problem 2. *Find the rotation matrix $R_{ab} = [\boldsymbol{x_{ab}} \quad \boldsymbol{y_{ab}}]$ for an arbitrary 2D rotation (as depicted in Fig. 1)*

$$R_{ab} = \begin{bmatrix} cos\theta & -sin\theta \\ sin\theta & cos\theta \end{bmatrix}$$

In general in 3D, we have three elemental rotation matrices that arise from rotations either about the x, y, or z-axis.

$$R_x(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & cos\theta & -sin\theta \\ 0 & sin\theta & cos\theta \end{bmatrix} ; \; R_y(\theta) = \begin{bmatrix} cos\theta & 0 & sin\theta \\ 0 & 1 & 0 \\ -sin\theta & 0 & cos\theta \end{bmatrix}$$

Problem 3. *Work out what $R_z(\theta)$ is.*

$$R_z = \begin{bmatrix} cos\theta & -sin\theta & 0 \\ sin\theta & cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

3 Uses of rotation matrices

3.1 Representing the orientation of a frame

This follows from the definition of a rotation matrix above. For any pair of coordinate frames A and B, there exists *one* unique rotation matrix R_{ab}— thus, R_{ab} tells us exactly how frame B is oriented from the reference of frame A.

Problem 4. *Find the rotation matrix R_{ba} given the same 2D coordinate frames in Fig. 1. What do you notice about the relationship between R_{ab} and R_{ba}?*

$$R_{ba} = \begin{bmatrix} cos\theta & sin\theta \\ -sin\theta & cos\theta \end{bmatrix}$$

We notice that R_{ab} and R_{ba} are the transpose of each other.

3.2 Associative Rule

Say we now have three frames A, B, and C. I tell you what R_{bc} is, (ie. the orientation of C from the reference of B), and you've already calculated R_{AB}. How can we express R_{ac}, that is, the orientation of C from the reference of A? We simply combine rotation matrices to form a new rotation matrix through matrix multiplication:

$$R_{ac} = R_{ab}R_{bc}$$

3.3 Changing the reference frame

Rotation matrices can also be used to represent motion.

Say we have a sensor on a robot arm at point q, which starts in the world origin frame A. The robot arm then rotates and ends up at a transformed frame B. The sensor is now still at coordinate q with respect to frame B, but we no longer know where it is with respect to the world frame. If we know the rotation matrix R_{AB}, how can we find where the point is with respect to the world frame?

$$q_A = R_{AB} q_B$$

Problem 5. *Given a point $q = (x, y)$, what are its new coordinates $q' = (x', y')$ after a rotation by a general θ counter-clockwise about the origin?*

$$x' = x cos\theta - y sin\theta$$

$$y' = x sin\theta + y cos\theta$$

4 Properties of rotation matrices

1. Columns of R are mutually orthonormal, ie. $RR^T = R^T R = I$
2. $det(R) = +1$ (right-handed coordinate frames)

Problem 6. *State whether the following matrix is a valid rotation:*

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

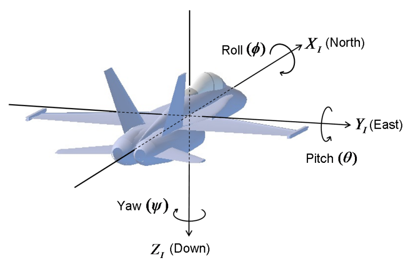
No. This matrix does not follow the right hand rule. The determinant is -1.

5 Other representations of rotations

5.1 RPY angles

Rotations are described by three angles (roll ϕ, pitch θ, yaw ψ) about the basis vectors of a **fixed coordinate frame** (say, the world frame). It involves the following intermediate rotations of a body frame B about a fixed world frame A, where B and A are initially coincident:

- Rotate A about its x-axis of A by the roll angle ϕ. Call this new frame B.
- Rotate B about the y-axis of A by the pitch angle θ. Call this updated frame C.
- Rotate C about the z-axis of A by the yaw angle ψ. Call this new and final frame D.



Thus, by the composition of these transformations, the final resultant rotation matrix is

$$R_{ad} = R_z(\psi)R_y(\theta)R_x(\phi)$$

This is an example of *extrinsic* rotations— ones that are all defined with respect to a fixed frame. Note that these are written right-to-left; that means the rightmost RPY operation is performed first, and the leftmost is performed last.

5.2 Euler angles

Euler angles describe the rotations about the **changing basis vectors**. For example, these are the rotations involved in what we call the ZYX Euler angles, here denoted as (α, β, γ). Let frame A be the initial orientation of the object being rotated.

- Rotate A by α about the z-axis. Call this new frame B.
- Rotate B by its *new* y-axis by β. Call this new frame C
- Rotate C by its *new* x-axis by γ. Call this new and final frame D.

The final resultant rotation matrix is derived from the composition of these rotations.

$$R_{ad} = R_{ab}R_{bc}R_{cd} = R_z(\alpha)R_y(\beta)R_x(\gamma)$$

This is an example of an *intrinsic* rotation — one that is defined with respect to the rotating coordinate frame. Note that these are written left-to-right; the leftmost is performed first.

5.3 Relationship between intrinsic and extrinsic rotations

It turns out that extrinsic rotation is equivalent to an intrinsic rotation by the same angles but with inverted order of elemental rotations, and vice-versa. Thus, a RPY transformation with roll, pitch, and yaw (XYZ) angles of (a, b, c) is equivalent to the (ZYX) Euler angle rotations of (c, b, a).

5.4 Quaternions

"Quaternions came from Hamilton after his really good work had been done; and, though beautifully ingenious, have been an unmixed evil to those who have touched them in any way..." — W. Thompson, Lord Kelvin. (1892).

The mathematics of quaternions is outside the scope of this course. However, unit quaternions are very useful for encoding 3D rotations, and you'll be seeing them a lot in lab. A unit quaternion Q is a vector with four components: x, y, z, and w, such that $\|Q\| = 1$. (Different software may represent quaternions as WXYZ or XYZW; watch out for that!) You can find these terms from an axis angle representation as follows:

$$w = \cos\left(\frac{\theta}{2}\right) \quad x = \omega_1 \sin\left(\frac{\theta}{2}\right) \quad y = \omega_2 \sin\left(\frac{\theta}{2}\right) \quad z = \omega_3 \sin\left(\frac{\theta}{2}\right)$$

where ω is a unit vector along the axis of rotation, and θ is the angle of rotation.

Benefits over Euler angles

- Represent SO(3) without singularities

Benefit over Rotation Matrices

- Only requires four values, rather than 9.
- Quaternion multiplication is much faster than matrix multiplication.

6 Rodrigues' formula

What if we don't want to use the elemental rotation matrices R_x, R_y, R_z? To express a general rotation about some axis ω with $||\omega|| = 1$ by some angle θ, we utilize Rodrigues' formula to extract the resulting rotation matrix:

$$R = I + \hat{\omega} sin\theta + \hat{\omega}^2(1 - cos\theta)$$

where the $\hat{}$ operator transforms a vector into its skew symmetric matrix as such:

$$\omega = \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix}; \quad \hat{\omega} = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix};$$