Practice Midterm 1

EECS/BioE C106A/206A
Introduction to Robotics

Due: October 1, 2020

Problem	**Max. Score**
Short Answers	20
Order of Operations	10
ROS Turtlesim Wrapper	15
Reference Frames	10
Forward Kinematics	8
Inverse Kinematics	12
When all else fails	15
Total	90

Problem 1. Short Answers (20 points)

(a) (5 points) Show that if $A \in \mathfrak{so}(n)$ is a skew-symmetric matrix then $R = e^A \in SO(n)$.

(b) (5 points) Let $g = (R, p) \in SE(3)$ be such that $R = R_Z(\pi)$ and $p = (0, 2, 0)$. Find a set of exponential coordinates for g.

(c) (5 points) Show that $R_X(\theta_1)R_X(\theta_2) = R_X(\theta_1 + \theta_2)$.

Hint: You may use the fact that for any square matrices A, B, if $AB = BA$ then $e^A e^B = e^{A+B}$.

(d) (5 points) Your friend from MIT asserts that for $a \in \mathbb{R}^3$, the matrix $B = (I - \hat{a})^{-1}(I + \hat{a})$ is in $SO(3)$; ie. B is a rotational matrix. True or False.

Hint 1: Don't try to brute-force this.
Hint 2: Remember the properties of skew-symmetric matrices
Hint 3: Does $(I - \hat{a})(I + \hat{a}) = (I + \hat{a})(I - \hat{a})$?

Problem 2. Order of Operations (10 points)

(a) (3 points) Select all operations that are always commutative:

- ☐ Multiple rotation matrices, about orthogonal axes
- ☐ Multiple rotation matrices, about parallel axes
- ☐ Multiple homogenous transforms, where all $R = I$
- ☐ Multiple homogenous transforms, where all $R = R_X(\frac{\pi}{4})$
- ☐ Multiple exponential mappings, with parallel revolute joints.
- ☐ Multiple exponential mappings, with parallel prismatic joints.

(b) (3 points) Select all options that are always associative:

- ☐ Multiple rotation matrices, about orthogonal axes
- ☐ Multiple rotation matrices, about parallel axes
- ☐ Multiple homogenous transforms, where all $R = I$
- ☐ Multiple homogenous transforms, where all $R = R_X(\frac{\pi}{4})$
- ☐ Multiple exponential mappings, with parallel revolute joints.
- ☐ Multiple exponential mappings, with parallel prismatic joints.

(c) (4 points) Answer the following questions.

(i) You are given the rotation matrices: R_{AB}, R_{CB}. Write an expression for R_{CA}.

(ii) You are given the rotation matrices: R_{AB}, R_{CA}. Write an expression for R_{BC}.

(iii) You are given the rotation matrices: R_{AB}, R_{BC}. Write an expression for R_{AA}.

(iv) You are given the rotation matrices: R_{AB}^{-1}, R_{BC}^{T}. Write an expression for R_{AC}.

Problem 3. Turtle Wrapper Node (15 points)

In Lab 1, we asked you to write a publisher node that would send a `geometry_msgs/Twist` over the `/turtle1/cmd_vel` topic in order to control your simulated turtle. You may recall only having to set two values of the twist: the linear x velocity, and the angular z velocity. This made sense at the time because our robot was entirely simulated in a 2D environment, reflecting the fact that it was a *unicycle modeled* robot; at any time, we may model the turtle's velocity relative to its own reference frame as

$$\vec{V} = \begin{bmatrix} v \\ \omega \end{bmatrix}.$$

where v is the linear x velocity and ω is the angular velocity. For a unicycle modeled robot, we always assume that the linear y velocity is 0. By controlling the turtle through directly manipulating a 6D Twist, you were breaking the abstraction between the perceived model of the robot and the commands the simulator needed to receive in order to control the turtle! To remedy this, you are now tasked with writing a "wrapper" node: you will construct a node that will listen for linear velocity and angular velocity commands published over a topic of your choice, and will publish that information to `/<turtle_name>/cmd_vel`, where `<turtle_name>` denotes the name of the turtle you want to control. Assume that your wrapper node has access to the desired turtle name through a command line argument. Assume this node will be written as a `.py` file placed in the appropriate location of a package named `midterm_1`.

(a) (3 points) What is the name of your node, what topic(s) do you want it to subscribe to, and what topic(s) do you want it to publish to? Remember that you want to be able to run multiple instances of your node if someone wants to use your wrapper node for multiple turtles.

(b) (2 points) You will be designing a new message type for the topic you choose to subscribe to. Define your `.msg` file. Make sure to indicate the name of the file somewhere.

(c) (5 points) Assume someone wants to control the turtle named "jturtle". A node named `user_control` is running that will send data of the appropriate message type to the topic your wrapper node subscribes to. Assuming that your wrapper node, turtlesim, and a `rostopic echo` node listening to the output of `user_control` for debugging purposes are all running. Draw the an approximate RQT graph that fits this scenario.

(d) (5 points) Time to code up your node! Fill in the appropriate blanks:

```python
#!/bin/env python

import rospy
import sys
from geometry_msgs.msg import Twist
from __________ import __________

class TurtleWrapper:
```

```
    def __init__(self, turtle_name):
        rospy.Subscriber(__________, __________, receive_command)
        self.pub = rospy.Publisher(__________, __________, queue_size=10)

    def receive_command(self, cmd_vel_2D):
        cmd_vel = Twist()
        cmd_vel.linear.x = __________
        cmd_vel.angular.z = __________
        self.pub.publish(cmd_vel)

if __name__ == '__main__':
    rospy.init_node(_________, anonymous=True)
    turtle_name = sys.argv[1]
    wrapper = TurtleWrapper(turtle_name)
    rospy.spin()
```

Problem 4. Reference Frames (10 points)

Figure 1 shows four reference frames in the workspace of a robot: the fixed frame $\{a\}$, the end-effector frame $\{b\}$, the camera frame $\{c\}$, and the workpiece frame $\{d\}$.

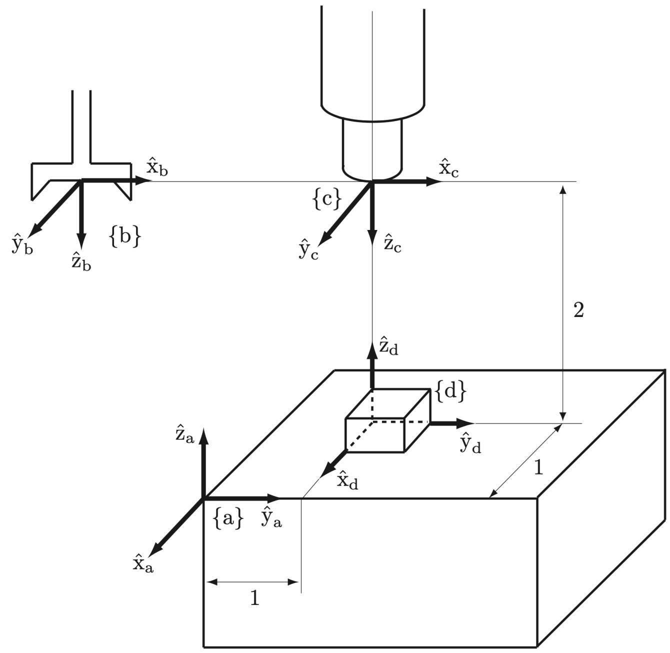

Figure 1: Four reference frames defined in a robot's workspace.

(a) (6 points) Find the $SE(3)$ poses g_{ad} and g_{cd} in terms of the dimensions given in the figure.

(b) (4 points) Find g_{ab} given that

$$g_{bc} = \begin{bmatrix} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Problem 5. Forward Kinematics (8 points)

Solve for the forward kinematics map of the 4DOF manipulator shown in the appendix in its initial configuration. The robot has three revolute and one prismatic joint. Do this by finding:

(a) (2 points) The initial configuration $g_{WT}(0) \in SE(3)$ of the robot.

(b) (4 points) The twists $\xi_1, \xi_2, \xi_3, \xi_4$ corresponding to each joint of the robot.

(c) (2 points) An expression for the forward kinematics map $g_{WT}(\theta)$ in terms of the vector of joint angles $\theta = (\theta_1, \theta_2, \theta_3, \theta_4)$. You may leave your answer in terms of the exponentials and products of known matrices.

Problem 6. Inverse Kinematics (12 points)

Consider the 4DOF manipulator shown in the appendix. Assume that $0 \leq \theta_4 \leq d_{max}$.

(a) (3 points) Describe the reachable workspace of the manipulator, that is the subset of $\mathbb{R}^3$ that the origin of the tool frame can reach. Ignore any self-collisions.

(b) (7 points) Use the Paden Kahan sub-problems to solve the inverse kinematics of this manipulator. You do not need to do the details of the inverse kinematics, but indicate how you would break down the inverse kinematics to get the angles.

(c) (2 points) Indicate the number of possible inverse kinematics solutions.

Problem 7. When all else fails (15 points)

Let $u \in \mathbb{R}^3$ be a unit vector, and let $R = I + 2\hat{u}^2$.

(a) (3 points) Show that $R^T R = I$.

Hint: Recall equation (2.13) from the textbook: $\hat{u}^3 = -\hat{u}$.

(b) (3 points) Show that $\det R = 1$, and hence conclude that R is a rotation matrix.

Hint: The function $u \mapsto \det(I + 2\hat{u}^2)$ *is continuous, thanks to the continuity of the determinant. However, from part (a) it follows that* $\det(I + 2\hat{u}^2) \in \{+1, -1\}$. *Is it possible for a continuous function to take on exactly two discrete values? Can you conclude from this that* $\det(I + 2\hat{u}^2)$ *is actually a constant, for any* u*?*

(c) (5 points) Find the exponential coordinates for R i.e. find a unit vector ω and a scalar $\theta \in [0, 2\pi)$ such that $R = e^{\hat{\omega}\theta}$.

Hint: What does R *look like when we switch to a new reference frame where* u *is the* x *axis?*

Hint: Recall that the determinant of a transformation is invariant under a change of basis.

(d) (2 points) Verify that when Rodrigues' formula is applied to your answer in the previous part, you get the original matrix R.

(e) (2 point) What would go wrong if you tried to use Proposition 2.9 from the textbook to compute the exponential coordinates of a rotation matrix of this form?

Appendix

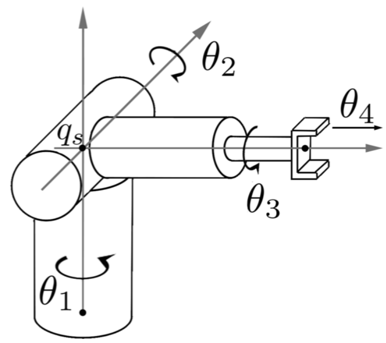

Figure 2: 4DOF Arm. Joints 1,2,3 are revolute. Joint 4 is prismatic

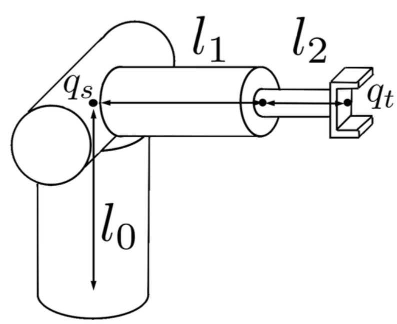

Figure 3: Manipulator lengths in zero configuration

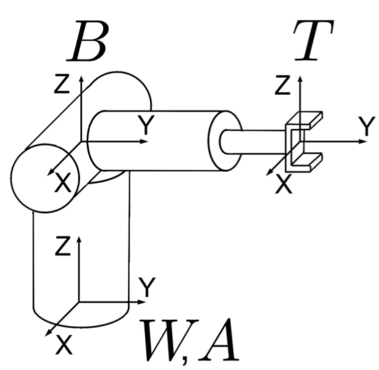

Figure 4: Coordinate Axes at zero configuration