Lost 1019

Wednesday, October 19, 2022 5:11 PM

LOST SECTION: 10/19

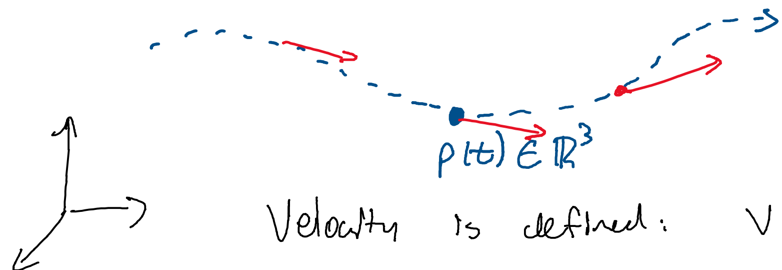
Today's Agenda:

1. Rigid body velocities
2. How to easily find rigid velocities
3. The Adjoint

Rigid Body Velocities:

- let's examine motion of a particle:

$$p(t) = \begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix} \qquad V(t) = \begin{bmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \\ \frac{dz}{dt} \end{bmatrix}$$



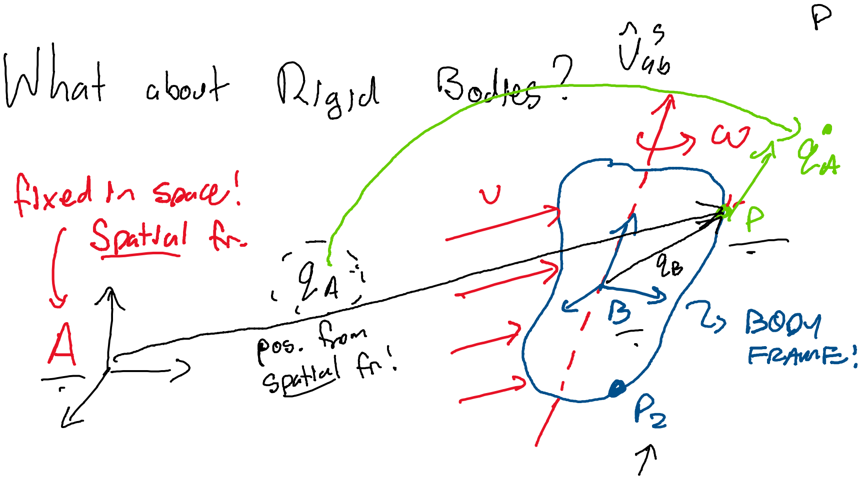
Velocity is defined: $V(t) = \dfrac{dp}{dt} = \dot{p}(t)$

Velocity: $\dfrac{\Delta p}{\Delta t} \Rightarrow \dfrac{m}{s}$

"dot" = time deriv!

$\ddot{p} \rightarrow$ 2nd deriv. wrt. t!

What about Rigid Bodies?



⇒ Two components!

- ANGULAR vel. ω
  ⇒ how quickly we spin
- ALSO have a lin. vel!

p_2 has a diff velocity!

⇒ Distance to axis affects speed!

- How do we describe the vel. of ANY pt. on a rigid body?
- Write out rel. btwn q_A & q_B in homog. coords:

$\dot{p}$

$$q_A = g_{ab}\, q_B$$

⇒ Take deriv!

$$\dot{q}_A = \dot{g}_{ab} q_B + \cancel{g_{ab} \dot{q}_B}$$ } "product rule"

constant!!

$$\dot{q}_A = \dot{g}_{ab} q_B$$

⇒ How can we find a MAP from a POINT in the spatial fr. to its vel. in the spatial frame??

$$q_A = g_{ab} q_B$$

$$g_{ab}^{-1} q_A = q_B \qquad \dot{q}_A = \dot{g}_{ab} q_B$$

⇒ Subst: $\boxed{\dot{q}_A = \dot{g}_{ab} g_{ab}^{-1} q_A}$

velocity of the pt. in the spatial fr!! — some transf! — point in spatial fr.

$$\hat{V}_{ab}^s = \dot{g}_{ab} g_{ab}^{-1}$$ } It is a TRANSF. from pos. to vel!

$$\dot{q}_A = \hat{V}_{ab}^s q_A$$

What do we know about V_{ab}^s?

Expand $\hat{V}_{ab}^s = \dot{g}_{ab} g_{ab}^{-1}$ $\qquad g_{ab} = \begin{bmatrix} R_{ab} & p_{ab} \\ 0 & 1 \end{bmatrix}$

$$\hat{V}_{ab}^s = \begin{bmatrix} \dot{R}_{ab} & \dot{p}_{ab} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} R_{ab}^T & -R_{ab}^T p_{ab} \\ 0 & 1 \end{bmatrix}$$ ⇒ Just take deriv. "element-wise!"

⇒ Mult out & check the structure!

$$\hat{V}_{ab}^s = \begin{bmatrix} \dot{R}_{ab} R_{ab}^T & -\dot{R}_{ab} R_{ab}^T p_{ab} + \dot{p}_{ab} \\ 0 & 0 \end{bmatrix} \in \mathbb{R}^{4\times4}$$

3x1 vector!!

3x3 matrix!

What does this look like? EXACTLY like hat map of a twist!

3x3 3x1

$$\hat{\xi} = \begin{bmatrix} \hat{\omega} & v \\ 0 & 0 \end{bmatrix} \text{ \} ALL ZEROS!}$$

$$\xi = \begin{bmatrix} v \\ \omega \end{bmatrix}$$

Big idea: ALL of the info about rigid body vel. is ENCODED in this 6x1 vec. V_{ab}^s!

$$(\hat{\omega})^\vee = \omega$$

Turn $\hat{V}_{ab}^s$ into a vector!

$$V_{ab}^s = \begin{bmatrix} -\dot{R}_{ab} R_{ab}^T p_{ab} + \dot{p}_{ab} \\ (\dot{R}_{ab} R_{ab}^T)^\vee \end{bmatrix} = \begin{bmatrix} v_{ab}^s \\ \omega_{ab}^s \end{bmatrix} \in \mathbb{R}^6$$

3x1 vec

3x1 vec!

$$\omega = \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix} \Rightarrow \hat{\omega} = \begin{bmatrix} 0 & - & - \\ + & 0 & + \\ + & - & 0 \end{bmatrix}$$

"Skew summ. matrices"

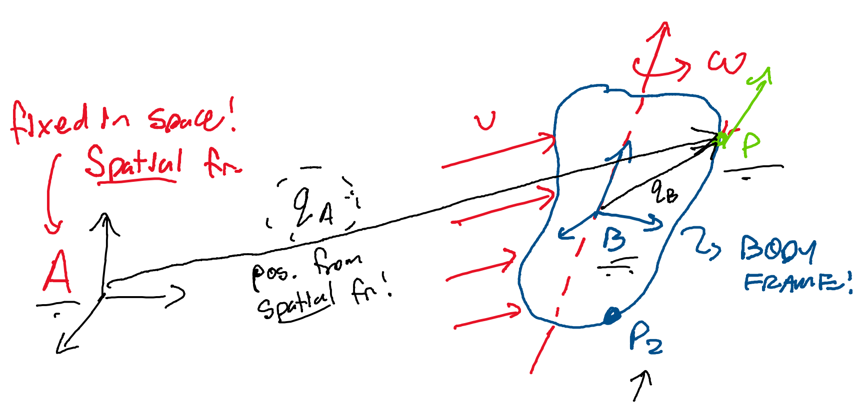
$$\hat{\omega} q = \omega \times q$$

Summary: we found $V_{ab}^s \Rightarrow$ "rigid body velocity"

$$\star \quad \dot{q}_A = \hat{V}_{ab}^s q_A \quad \star$$

$$\hat{V}_{ab}^s = \dot{g}_{ab} g_{ab}^{-1}$$

BODY VELOCITY:



$$\dot{q}_A = \dot{g}_{ab} g_{ab}^{-1} q_A$$

} try and take $\dot{q}_A$ from the spatial fr. to $[\dot{q}_A]_B \Rightarrow$ its repr. in the BODY fr.!

$$[\dot{q}_A] = g^{-1} \dot{q}_A$$

$[q_A]_B = g_{ab}\, q_A$ ... for body fr.

$\Rightarrow$ Try and find a map from q_B to $[\dot{q}_A]_B$ "spatial vel. as seen from B"

$$[\dot{q}_A]_B = g_{ab}^{-1}\,\dot{g}_{ab}\,g_{ab}^{-1}\,q_A$$

$$[\dot{q}_A]_B = g_{ab}^{-1}\,\dot{g}_{ab}\,\cancel{g_{ab}^{-1}}\,(\cancel{g_{ab}}\,q_B)$$

$$\boxed{[\dot{q}_A]_B = g_{ab}^{-1}\,\dot{g}_{ab}\,q_B}$$

spatial vel. as seen from B! — point in fr. B

$$\hat{V}_{ab}^{b} = g_{ab}^{-1}\,\dot{g}_{ab}$$

$\hat{V}_{ab}^{s}$ } NOT a velocity! $\Rightarrow$ Is a TRANSF. that helps us find velocity!

☆ $\boxed{[\dot{q}_A]_B = \hat{V}_{ab}^{b}\, q_B}$ ☆

$$\hat{V}_{ab}^{b} = \begin{bmatrix} \hat{\omega}_{ab}^{b} & v_{ab}^{b} \\ 0 & 0 \end{bmatrix} \Rightarrow \boxed{V_{ab}^{b} = \begin{bmatrix} v_{ab}^{b} \\ \omega_{ab}^{b} \end{bmatrix}}$$

Rigid body "BODY" vel!

Summary: components of un vel!

$$\hat{V}_{ab}^{s} = \dot{g}_{ab}\,g_{ab}^{-1}$$

$$V_{ab}^{s} = \begin{bmatrix} v_{ab}^{s} \\ \omega_{ab}^{s} \end{bmatrix} \qquad \hookrightarrow V_{ab}^{s} = \begin{bmatrix} v_{ab}^{s} \\ \omega_{ab}^{s} \end{bmatrix}$$

"ACTUAL" orig vel! $V_{q_A} = \dot{q}_A = \hat{V}_{ab}^{s}\, q_A$

} SPATIAL vel!

$\hat{V}$ = "MAP FROM A PT TO ITS VEL"

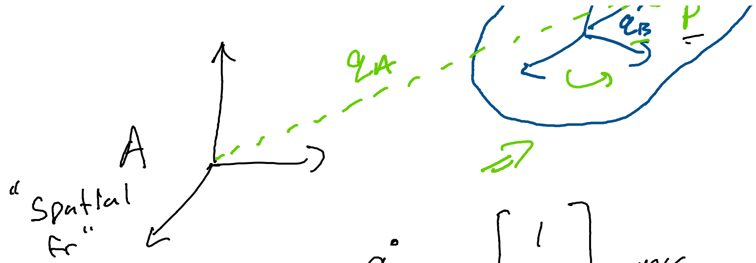
$$\hat{V}_{ab}^{b} = g_{ab}^{-1}\,\dot{g}_{ab}$$

$$\hookrightarrow V_{ab}^{b} = \begin{bmatrix} v_{ab}^{b} \\ \omega_{ab}^{b} \end{bmatrix}$$

$V_{q_B} = [\dot{q}_A]_B = \hat{V}_{ab}^{b}\, q_B$ } "BODY" vel!

NOT $= \dot{q}_B$!!

$\dot{q}_A \neq 0$ "spatial vel"



$$\dot{q}_B = 0$$

$$V_{qB} = [\dot{q}_A]_B \qquad [V]_B$$

$$\dot{q}_A = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \text{ m/s}$$

$$R_{ba} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$$V_{qB} = R_{ba}\,\dot{q}_A$$

$$V_{qB} = [\dot{q}_A]_B = \begin{bmatrix} 1 \\ -2 \\ -3 \end{bmatrix}$$

$$R_z(\theta) = \begin{bmatrix} \cos\theta(t) & \sin\theta(t) & 0 \\ -\sin\theta(t) & \cos\theta(t) & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \dot{R}_z = \begin{bmatrix} -\dot{\theta}\sin\theta & \cdots \\ -\dot{\theta}\cos\theta & \cdots \\ \cdots & \cdots \end{bmatrix}$$

Finding + Interpreting Rigid Body Vels:

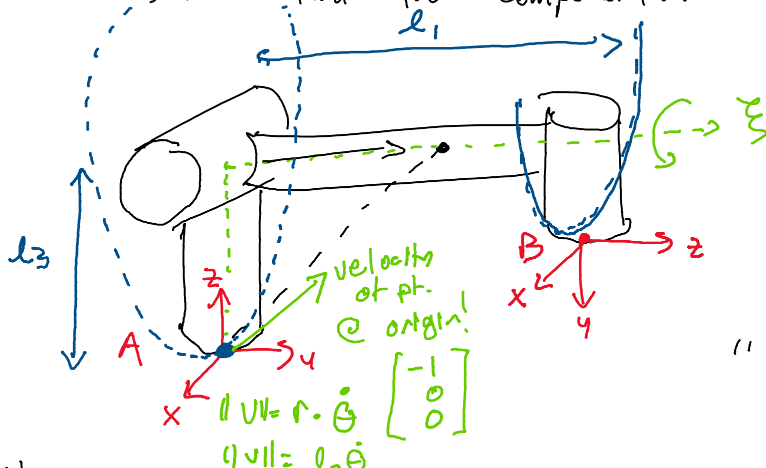
$$\hat{V}^s_{ab} = \dot{g}_{ab}\, g^{-1}_{ab}$$
$$\hat{V}^b_{ab} = g^{-1}_{ab}\, \dot{g}_{ab}$$

Are they practical?? NO!!

⇒ SOME SHORTCUT that helps us find these vecs!

Q: How can we find $V^s_{ab} = \begin{bmatrix} v^s_{ab} \\ \omega^s_{ab} \end{bmatrix}$??

⇒ Must find the components!

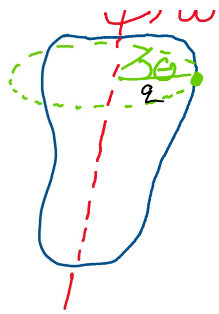


Q: Find V^s_{ab}, V^b_{ab} GIVEN $\dot{\theta}$!

$$V^s_{ab} = \begin{bmatrix} v^s_{ab} \\ \omega^s_{ab} \end{bmatrix}$$

"CIRCLE METHOD"

How can we simplify our solution??



⇒ Spins on axis ω

⇒ Spins w/ angular speed $\|\omega\| = \dot{\theta}$

$v = \omega \times q$

↑ dist. to axis!

Actual rate of rotation!

$$\dot{\bar{q}}_A = \hat{V}^s_{ab}\, \bar{q}_A$$

$$\dot{\bar{q}}_A = \begin{bmatrix} \hat{\omega}^s_{ab} & v^s_{ab} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} q_x \\ q_y \\ q_z \\ 1 \end{bmatrix} \quad \text{"}q_A\text{"} \qquad \frac{d}{dt}\begin{bmatrix} q_A \\ 1 \end{bmatrix} = \begin{bmatrix} \dot{q}_A \\ 0 \end{bmatrix}$$

no longer in homog!

$$\begin{bmatrix} \dot{q}_A \\ 0 \end{bmatrix} = \begin{bmatrix} \hat{\omega}^s_{ab} q_A + v^s_{ab} \\ 0 \end{bmatrix}$$

$$\dot{q}_A = \underline{\omega^s_{ab} \times q_A} + \underline{v^s_{ab}}$$

⇒ ω^s_{ab} is JUST the spatial angular velocity vector!!

Here: ω^s_{ab} ⇒ in dir. $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, w/ mag. $\dot{\theta}$

$$\underline{\omega^s_{ab}} = \begin{bmatrix} 0 \\ \dot{\theta} \\ 0 \end{bmatrix} \qquad V^s_{ab} = \begin{bmatrix} v^s_{ab} \\ \omega^s_{ab} \end{bmatrix}$$

What is v^s_{ab}??

$$\dot{q}_A = \overbrace{\omega^s_{ab} \times q_A}^{\text{"rotational"}} + \overbrace{v^s_{ab}}^{\text{"transl"}}$$

Drop out so we can get v^s_{ab} on its own!

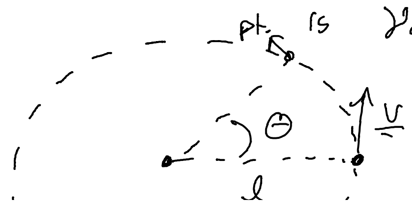
⇒ Focus on $\underline{q_A = 0}$

$$\dot{q}_A = v^s_{ab}$$

When we look at the motion of a point passing through ORIGIN of the spatial fr, the vel. of that pt. is v^s_{ab}!!

What is v^s_{ab}?

⇒ dir: $\begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}$



$\|v\| = l\dot{\theta}$

$\dot{\theta} = \frac{1}{c}$ $l = n$

$$\|v_{ab}^s\| = l_3\dot{\theta}$$

units $\frac{?}{}$
$m \cdot \frac{1}{s}$
$= \frac{m}{s}$

$$v_{ab}^s = \begin{bmatrix} -l_3\dot{\theta} \\ 0 \\ 0 \end{bmatrix}$$

$$\boxed{\underline{V}_{ab}^s = \begin{bmatrix} v_{ab}^s \\ \omega_{ab}^s \end{bmatrix} = \begin{bmatrix} -l_3\dot{\theta} \\ 0 \\ 0 \\ 0 \\ \dot{\theta} \\ 0 \end{bmatrix} \begin{matrix} \}\, v_{ab}^s \\ \}\, \omega_{ab}^s \end{matrix}}$$

NOT actual vel!

For ANY $\dot{q}_A = \underline{\hat{V}}_{ab}^s q_a$

$$\hat{V}_{ab}^s = \begin{bmatrix} \hat{\omega}_{ab}^s & v_{ab} \\ 0 & 0 \end{bmatrix}$$

$\Rightarrow$ Transf. for ANY pt. to its vel!

BODY VELOCITY:

Transf. that gives us this for ANY pt. in Body fr.

$$[\dot{q}_A]_B = \underline{\hat{V}}_{ab}^b q_B$$

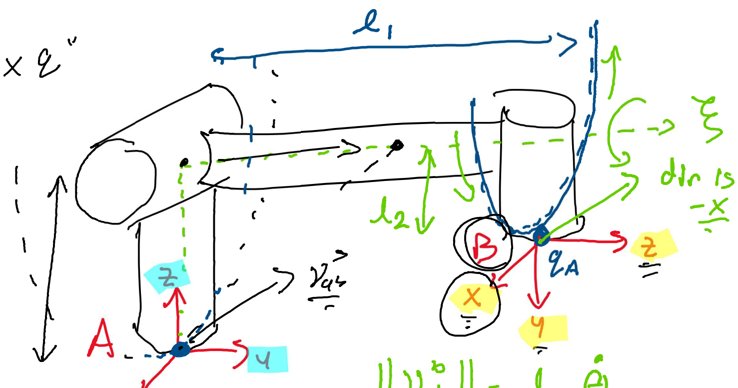
"Spatial vel as seen from BODY fr."

$$[\dot{q}_A]_B = \hat{V}_{ab}^b \bar{q}_B$$

$$\begin{bmatrix} [\dot{q}_A]_B \\ 0 \end{bmatrix} = \begin{bmatrix} \hat{\omega}_{ab}^b & v_{ab}^b \\ 0 & 0 \end{bmatrix} \begin{bmatrix} q_B \\ 1 \end{bmatrix}$$

"$\omega_{ab} \times q$"

$\omega \times r$



$$\|v_{ab}^b\| = l_2\dot{\theta}$$

$$[\dot{q}_A]_B = \underline{\omega_{ab}^b \times q_B} + v_{ab}^b$$

What is ω_{ab}^b??

$\omega \times q$

ω_{ab}^b is the ANGULAR VEL. VEC. AS SEEN from the BODY fr!

$$\|\omega_{ab}^b\| = \dot{\theta}$$

$\Rightarrow$ Dir. is WRT BODY fr!!

$$\text{dir. of } \omega_{ab}^b = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\omega_{ab}^b = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

} Angular vel. vec. IN the body fr!

$[\dot{\theta}] \quad \}$

Let's solve for v_{ab}^b !

$[\dot{q}_A]_B = \underline{\omega_{ab}^b \times q_b} + \underline{v_{ab}^b}$ } Holds for ANY pt. in body fr!

"Choose" $q_b = 0$ => origin of body frame!!

$[\dot{q}_A]_B = \underline{v_{ab}^b}$ } v_{ab}^b is the Spatial VELOCITY of the body frame, transformed to the body fr:

direc: $\begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}$

$v_{ab}^b = \begin{bmatrix} -l_2\dot{\theta} \\ 0 \\ 0 \end{bmatrix}$

mag: $\|v_{ab}^b\| = l_2\dot{\theta}$

$$\boxed{V_{ab}^b = \begin{bmatrix} v_{ab}^b \\ \omega_{ab}^b \end{bmatrix} = \begin{bmatrix} -l_2\dot{\theta} \\ 0 \\ 0 \\ 0 \\ 0 \\ \dot{\theta} \end{bmatrix}}$$

"BODY VELOCITY"

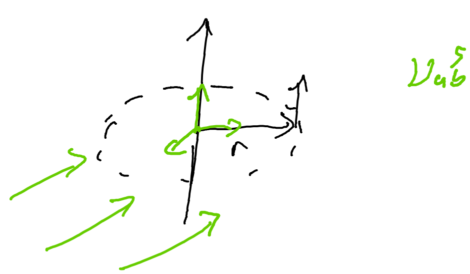
Summary:

Pick a point q_A in the spatial fr:

Its vel. is: $\underline{\dot{q}_A = \hat{V}_{ab}^s q_A}$ $\quad \hat{V}_{ab}^s = \widehat{\begin{bmatrix} v_{ab}^s \\ \omega_{ab}^s \end{bmatrix}}$

=> If we pick a point q_B in the BODY fr,

Its SPATIAL vel. from the body fr. is: $\underline{[\dot{q}_A]_B = \hat{V}_{ab}^b q_B}$



Adjoint: $V_{ab}^s \quad V_{ab}^b$ "IS THERE A MATRIX ... TRANSFER?"

$\in \mathbb{R}^6$, $\in \mathbb{R}^6$

that helps us transf...

⇒ Can't just apply g

⇒ Will need a 6x6 matrix!

"Adjoint of g" ⇒ Transforms rigid body vels!

$$\boxed{V_{ab}^s = Ad_{g_{ab}} V_{ab}^b}$$

Ad_g ACTS ON VECTORS!!

$$g_{ab} = \begin{bmatrix} R_{ab} & p_{ab} \\ 0 & 1 \end{bmatrix}$$

$$Ad_{g_{ab}} = \begin{bmatrix} R_{ab} & \hat{p}_{ab} R_{ab} \\ 0 & R_{ab} \end{bmatrix}$$

⇒ Makes it v. easy to transf. btwn. velocities!

$$\left.\begin{aligned} \hat{V}_{ab}^s &= \dot{g}_{ab} g_{ab}^{-1} \\ \hat{V}_{ab}^b &= g_{ab}^{-1} \dot{g}_{ab} \end{aligned}\right\}$$

look for a way to relate these!

"Conjugate transpose adjoint"

"ADJOINT" ⇒ useful & related to our original!

$$\hat{V}_{ab}^s = g_{ab} (g_{ab}^{-1} \dot{g}_{ab}) g_{ab}^{-1}$$

$$\boxed{\hat{V}_{ab}^s = g_{ab} \hat{V}_{ab}^b g_{ab}^{-1}}$$

$$\boxed{V_{ab}^s = (g_{ab} \hat{V}_{ab}^b g_{ab}^{-1})^\vee = Ad_{g_{ab}} V_{ab}^b}$$

$$(Ad_{g_{ab}} V_{ab}^b)^\wedge = g_{ab} \hat{V}_{ab} g_{ab}^{-1}$$