# LOST 0914

Wednesday, September 14, 2022 5:09 PM

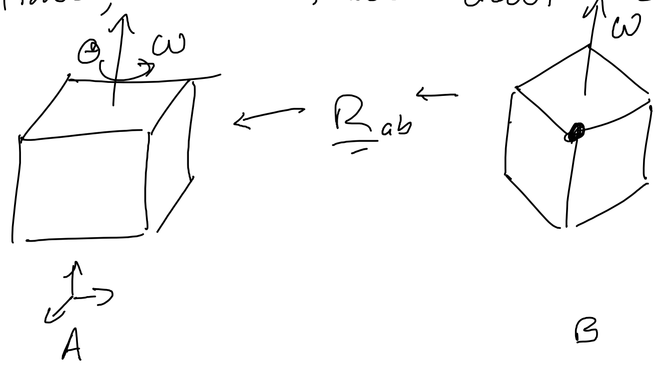
## LOST SECTION: 9/14

**Agenda:**

1. General Rigid Body Motions & SE(3)
2. Exponential Coords
3. Screw Motion!

General Rigid Motions:

- last time, we talked about SO(3):



$SO(3) \Rightarrow$ Represent <u>ANY</u> rotation using a matrix $R \in SO(3)$

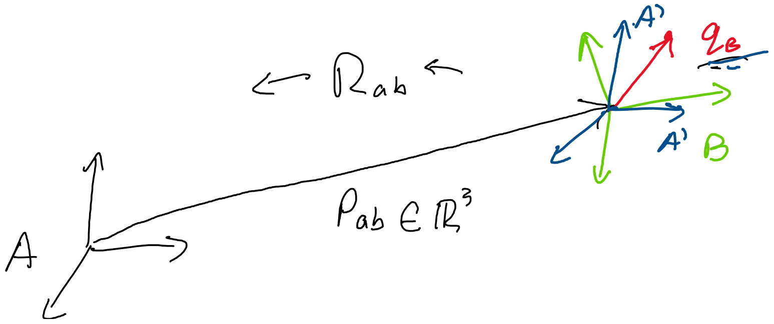
$\in$

How do we deal with TRANSLATIONS + Rotations?

In general, a Rigid body transf. is made up of:

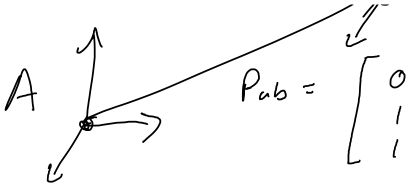
1. Rotation: $R \in SO(3)$
2. Translation: $p \in \mathbb{R}^3$

Coord. Frames & Rigid Motions:



$q_{A'} = \begin{bmatrix}0\\0\\0\end{bmatrix}_{A'}$

$$q_{A'} = \underline{R_{ab}\, q_B}$$



$$P_{ab} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

Add the translation:

$$q_A = q_{A'} + p_{ab}$$

$$q_A = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}_{A'} + \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

$$\boxed{q_A = R_{ab}\, q_B + p_{ab}}$$

$$q_A = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

Represent: $q_B \rightarrow q_A$ as a SINGLE matrix!

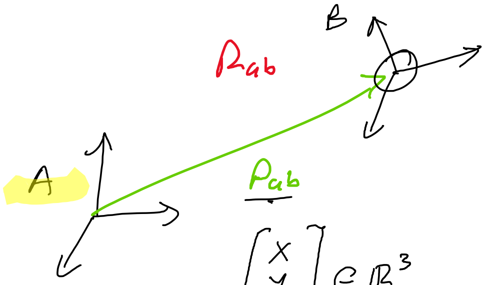
$$\begin{bmatrix} q_A \\ 1 \end{bmatrix} = \begin{bmatrix} R_{ab}\, q_b + \boxed{p_{ab}} \\ 1 \end{bmatrix} \qquad \begin{bmatrix} q_b \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} q_A \\ 1 \end{bmatrix} = \begin{bmatrix} R_{ab} & p_{ab} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} q_b \\ 1 \end{bmatrix}$$

"Homogeneous Coordinates"

$$p = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \qquad \bar{p} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 1 \end{bmatrix}$$

Overbar = homogeneous!



$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3$$

$$\bar{q}_A = \begin{bmatrix} R_{ab} & p_{ab} \\ 0 & 1 \end{bmatrix} \bar{q}_b$$

SO(3) → 3x3 rotation matrix

$$☆ \quad \boxed{\bar{q}_A = g_{ab}\, \bar{q}_b} \quad ☆$$

SUMMARY:

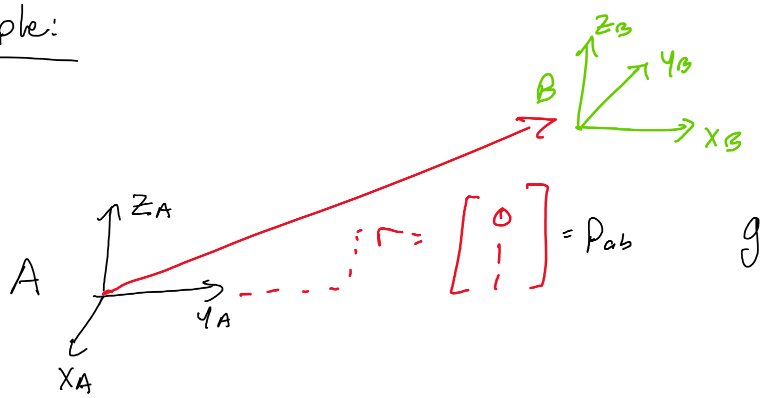
$$g = \begin{bmatrix} \overset{\in SO(3)}{R} & \overset{\in \mathbb{R}^3}{p} \\ 0 & 1 \end{bmatrix}$$

$$SE(3) = \left\{ \begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix}, \ \underset{\text{rotation matrix}}{R \in SO(3)}, \ \underset{\text{translation}}{p \in \mathbb{R}^3} \right\}$$

SE(3) is the Space of ALL general rigid body transforms.

Example:

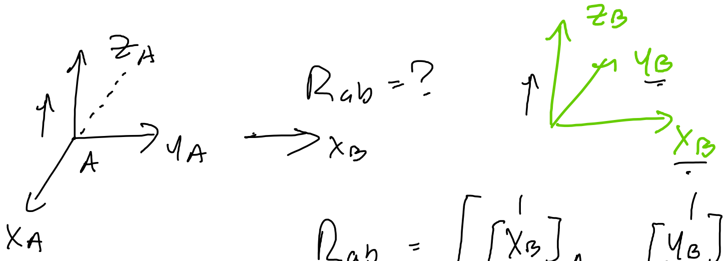
Undo



$r = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = p_{ab}$

$g_{ab} \in SE(3)$

$$g_{ab} = \begin{bmatrix} R_{ab}\,\checkmark & p_{ab}\,? \\ 0 & 1 \end{bmatrix}$$



$R_{ab} = ?$

$$R_{ab} = \begin{bmatrix} [x_B]_A & [y_B]_A & [z_B]_A \end{bmatrix}$$

$$R_{ab} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad p_{ab} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

$$g_{ab} = \begin{bmatrix} R_{ab} & p_{ab} \\ 0 & \textcircled{1} \end{bmatrix} \qquad \begin{bmatrix} \overset{\text{matrix}}{R} & p \\ 0 & 1 \end{bmatrix} \to \text{vector}$$

$$\boxed{g_{ab} = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \in \mathbb{R}^{4\times4}}$$

SE(3) -> 4x4 matrices!

## Summary:

Rotations and translations using matrices in SE(3):

$$g = \begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix} \in \mathbb{R}^{4\times4}$$

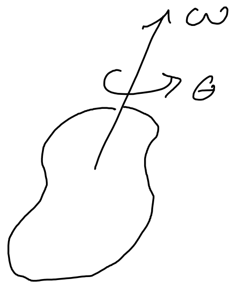
$$\boxed{\bar{q}_A = g_{ab}\,\bar{q}_b}$$

# TWISTS:

- After making a definition for SO(3) → What did we do next?
  ⇒ exp. coordinates!
  - $R \in SO(3)$ ⇒ represent with only ③ parameters!

$$R = \underline{e^{\hat{\omega}\theta}}$$

rotate about ω
by angle θ



## SE(3)?

$$\begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix} \in \mathbb{R}^{4\times4}$$

$\underline{16}$ total parameters!

Are there constraints??

- 3 rotational D.O.F.
- 3 translational DOF (x, y, z)

⑥ parameters to represent $g \in SE(3)$??

$$\underline{\omega} \in \mathbb{R}^3 \longrightarrow e^{\hat{\omega}\theta} = \underline{R} \in SO(3)$$

$$\underline{\xi \in \mathbb{R}^6} \rightarrow \underline{e^{\hat{\xi}\theta}} = \underline{g} \in SE(3)$$

$\underline{\xi}$ "$\underline{\text{Xi}}$" ⇒ "Z-eye" ξ ξ ξ ξ

ζ "$\underline{\text{zeta}}$"

"angular velocity" ⇒ 3D vector that descr. rotations

$\hat{\omega}$ = matrix of taking cross prod. with ω.

$\hat{\omega} r = \omega \times r$

$V = \omega \times r$

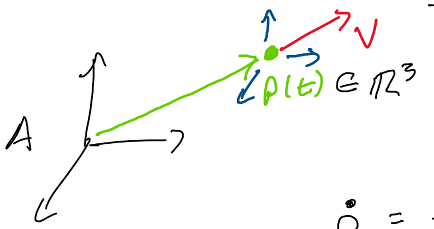
$\dot{r} = \omega \times r$

$\underline{\dot{r}} = \underline{\hat{\omega} r}$

$r(\theta) = e^{\hat{\omega}\theta} \cdot r(0)$

$\underline{\|\omega\| = 1}$

① Pure translation: $\underline{e^{\hat\xi\theta}}$ $\underline{e^{\hat\xi_1\theta_1}e^{\hat\xi_2\theta_2}\dots e^{\hat\xi_n\theta_n}}$



$\underline{\|v\| = 1}$ m/s

$$\dot p = \frac{dp}{dt} = v$$

$$\frac{d}{dt}\begin{bmatrix}p\\1\end{bmatrix} = \begin{bmatrix}v\\0\end{bmatrix}$$

$$\left.\begin{aligned}\begin{bmatrix}\dot p\\0\end{bmatrix} &= \begin{bmatrix}v\\0\end{bmatrix}\\ \dot{\bar p} &= \bar v\end{aligned}\right\}\ \underline{\text{MAP}}\ \bar p \to \dot{\bar p}$$

$$\begin{bmatrix}\dot p\\0\end{bmatrix} = \begin{bmatrix} & ? & \end{bmatrix}\begin{bmatrix}p\\1\end{bmatrix}$$

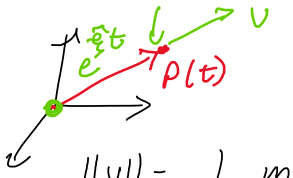
$$\begin{bmatrix}\dot p\\0\end{bmatrix} = \underset{\uparrow\ \underline{\text{MAP}}\ \text{from}\ \bar p\ \text{to}\ \dot{\bar p}}{\begin{bmatrix}0_{3x3} & v\\0 & 0\end{bmatrix}}\begin{bmatrix}p\\1\end{bmatrix}$$

$$\dot{\bar p} = \underset{\uparrow\ \hat\xi}{\begin{bmatrix}0_{3x3} & v\\0 & 0\end{bmatrix}}\bar p$$

$$\dot{\bar p} = \hat\xi\bar p$$

$$\underline{\bar p(t) = e^{\hat\xi t}\bar p(0)}$$

$$\boxed{\bar p(\theta) = e^{\hat\xi\theta}\bar p(0)}$$



$\underline{\|v\|=1}$

$\|v\| = 1$ m/s

For every 1 sec, we move 1 m.

θ ⇒ $\underline{\text{distance traveled}}$

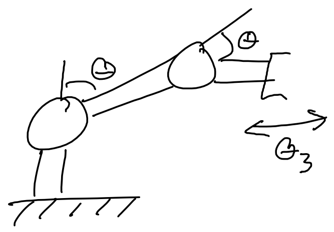
$e^{\hat\xi\theta}$ ⇒ transformation of MOVING distance θ In DIRECTION ⓥ

$\xi = \begin{bmatrix}v\\\omega\end{bmatrix}$

⇒ $\underline{V}$ is NOT the $\underline{\text{ACTUAL}}$ velocity of the body!

- $\underline{\text{Just}}$ a vec. POINTING where we want to go!

Undo



$$e^{\hat{\xi}_1\theta_1} = e^{\hat{\xi}_2\theta_2} = e^{\hat{\xi}_3\theta_3}$$

$$g = \begin{bmatrix} I & p \\ 0 & 1 \end{bmatrix}$$

$$v = \frac{p}{\|p\|}$$

Translational Case:

$e^{\hat{\xi}\theta}$ } translation of θ in direction v

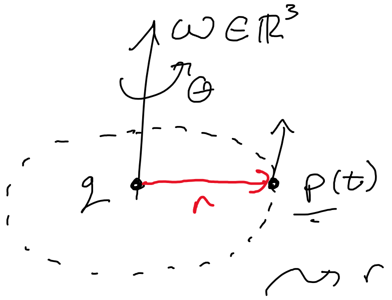
No rotation

$\rightsquigarrow p =$ tran of θ

$$\hat{\xi} = \begin{bmatrix} 0_{3\times3} & v \\ 0 & 0 \end{bmatrix} \rightsquigarrow \text{"direction" / "velocity"} \Rightarrow e^{\hat{\xi}\theta} = \begin{bmatrix} I_3 & v\theta \\ 0 & 0 \end{bmatrix}$$

Rotational Case: (NO TRANSLATION)

$\|\omega\| = 1$ rad/s



ω is an ANGULAR vel. vec.

$\|\omega\|$ = speed in radians/sec

dir → ω = AXIS of rotation!

$$\dot{p} = v = \underline{\omega \times (p - q)}$$

Want to find a MAP between $\bar{p}$ & $\dot{\bar{p}}$

$$\frac{d}{dt}\begin{bmatrix} p \\ 1 \end{bmatrix} = \begin{bmatrix} \omega \times (p-q) \\ 0 \end{bmatrix}$$

extract $\bar{p}$

$$\begin{bmatrix} \dot{p} \\ 0 \end{bmatrix} = \begin{bmatrix} \omega \times (p-q) \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} \dot{p} \\ 0 \end{bmatrix} = \begin{bmatrix} \underline{\hat{\omega} p} - \underline{\omega \times q} \\ 0 \end{bmatrix}$$

$\dot{\bar{p}} \rightsquigarrow$

$$\begin{bmatrix} \dot{p} \\ 0 \end{bmatrix} = \begin{bmatrix} ? \end{bmatrix} \underset{\bar{p}}{\begin{bmatrix} p \\ 1 \end{bmatrix}}$$

$$\begin{bmatrix} \dot{p} \\ 0 \end{bmatrix} = \begin{bmatrix} \hat{\omega} & -\omega \times q \\ 0 & 0 \end{bmatrix} \begin{bmatrix} p \\ 1 \end{bmatrix}$$

What is $-\omega \times q$??

$$\dot{p} = \underline{\omega \times p} - \underline{\omega \times q}$$

$\omega \times q$ is a COMPONENT of lin. velocity!

⇒ "What's left over"

$$\dot{\bar{p}} = \begin{bmatrix} \hat{\omega} & -\omega \times q \end{bmatrix} \bar{p}$$

Undo

$$\begin{bmatrix} \cdots & \cdots \\ 0 & 0 \end{bmatrix}$$

⇒ what's left after doing $\omega \times p$

$\hat{\xi}$ : <u>MAP</u> from $\bar{p}$ to $\dot{\bar{p}}$

$$\dot{\bar{p}} = \hat{\xi}\,\bar{p}$$

$\|\omega\| = 1$ rad/s

$\theta \leftrightarrow t$

↑

ANGLE OF ROTATION!

"$2\pi$ seconds"

$$\bar{p}(t) = e^{\hat{\xi}t}\,\bar{p}(0)$$

$$\boxed{\bar{p}(\theta) = e^{\hat{\xi}\theta}\,\bar{p}(0)}$$

Exp. coords. for ROTATION!

PURE TRANSL:

$$\hat{\xi} = \begin{bmatrix} 0_{3\times 3} & \overset{\text{dir. vec.}}{v} \\ 0 & 0 \end{bmatrix}$$

$v$ ⇒ vector pointing in direction of motion "velocity"

$\|v\| = 1$

$e^{\hat{\xi}\theta}$ ⇒ $g \in SE(3)$ of PURE translation!

PURE ROTATION:

$$\hat{\xi} = \begin{bmatrix} \hat{\omega} & \overset{\text{"component"}}{-\omega \times q} \\ 0 & 0 \end{bmatrix}$$

$\|\omega\| = 1$, $\omega$ was the <u>AXIS</u> of rotation! "angular velocity"

$e^{\hat{\xi}\theta}$ ⇒ $g \in SE(3)$ of PURE ROTATION

In GENERAL, for <u>ANY</u> rigid body motion:

$$\hat{\xi} = \begin{bmatrix} \hat{\omega} & v \\ 0 & 0 \end{bmatrix}$$

for transl, $\omega = 0$

Wanted ⑥ parameters for $g \in SE(3)$

$\hat{\omega} \rightarrow \omega \in \mathbb{R}^3$ for $SO(3)$

$\hat{\xi} \rightarrow \xi \in \mathbb{R}^6$ for $SE(3)$

| | NONZERO Rotation | Pure transl. |
|---|---|---|
| $\omega$ unit vec: | ✓ | X |
| $v$ unit vec: | X | ✓ |

$v = -\underline{\underline{\omega}} \times \underline{\underline{q}}$ ⇒ <u>Not</u> nec. a unit vec!

Extract essential info. from $\hat{\xi}$

$$\boxed{\xi = \begin{bmatrix} v \\ \omega \end{bmatrix} \in \mathbb{R}^6}$$

$v \in \mathbb{R}^3$

$\omega \in \mathbb{R}^3$ Sometimes 0, sometimes AXIS! (transl.)

$\hat{\xi}\theta$

TWIST: $(\xi, \theta) \Rightarrow g = e \in SE(3)$

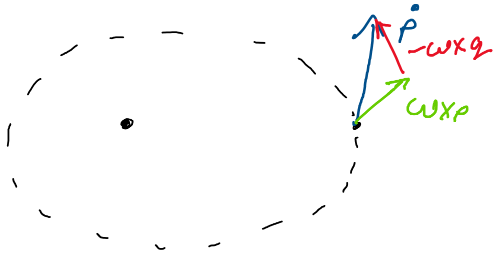
$\xi$ ↑ twist

$\theta$ ↑ "displacement" linear / angular

"EXP. COORDS. for SE(3)"

Nonzero rotation:

PURE rotation: $\dot{p} = \underbrace{\omega \times p}_{\hat{\omega} p} \underbrace{- \omega \times q}_{1}$ $V = -\omega \times q$



$g_r \cdot g_t = g_{rt}$

$\underbrace{e^{\hat{\xi}_r \theta_r} \cdot e^{\hat{\xi}_t \theta_t}} = g_{rt}(\theta_r, \theta_t)$

with a SINGLE $\hat{\xi}$ & a single $\theta$ "Screw"

Transl: $\overline{p}(\theta) = e^{\hat{\xi}\theta} p(0)$ $e^{\hat{\xi}\theta}$ { IS a rigid body transf. $g \in SE(3)$

↑ dist. $\theta$

↑ transformation g of SHIFTING by $\theta$ in direction v

$$e^{\hat{\xi}\theta} = \begin{bmatrix} I_3 & v\theta \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix}$$

Gen. case: $e^{\hat{\xi}\theta} = \begin{bmatrix} e^{\hat{\omega}\theta} & (I - \ldots) \ldots + \omega \omega^T v \theta \\ 0 & 1 \end{bmatrix}$

"Chasles Thm"

How to deal with Rotation & Translation:

⇒ Screw Motion!

Try and find $\xi \in \mathbb{R}^6$ s.t. $e^{\hat{\xi}\theta} = g \in SE(3)$

↙ only thing controlling distance!

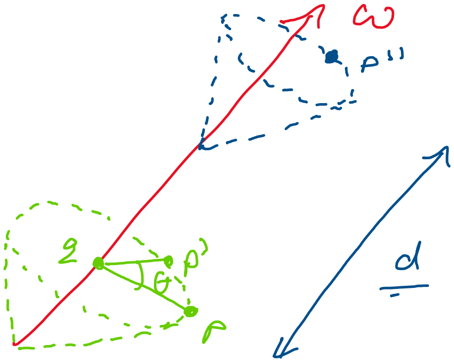
↑ ANYTHING!!!

SCREW MOTION:

SCREW MOTION:

1. Rotate by ANGLE $\theta$ about an axis $\omega$, $\|\omega\| = 1$
2. Translate ALONG the unit axis $\omega$ by distance $\underline{d}$.

What does this look like?



$\xi$??

What's the twist for the $\underline{\text{rotation}}$?

$$\xi_{rotation} = \begin{bmatrix} -\omega \times q \\ \omega \end{bmatrix}$$ ⇒ "left over" term in velocity after doing $\omega \times p = \hat{\omega} p$

$e^{\hat{\xi}_r \theta}$ ⇒ rotation by $\theta$ about the axis $\omega$.

$\underline{\text{Translation?}}$

$$\xi_t = \begin{bmatrix} v \\ 0 \end{bmatrix}$$

Must CONVERT btwn. $\theta$ (angle we rotate) & distance $\textcircled{d}$ we want to travel!!

$$\text{"PITCH"} \Rightarrow h = \frac{d}{\theta}$$

$$\xi_t = \begin{bmatrix} \frac{d}{\theta} \cdot \omega \\ 0 \end{bmatrix} = \begin{bmatrix} h\omega \\ 0 \end{bmatrix}$$

$$\xi_t \cdot \theta = \begin{bmatrix} \frac{d}{\cancel{\theta}} \cdot \omega \cdot \cancel{\theta} \\ 0 \end{bmatrix} = \begin{bmatrix} d\omega \\ 0 \end{bmatrix}$$

$$\boxed{\xi = \xi_t + \xi_r = \begin{bmatrix} -\underline{\omega} \times q + h\underline{\omega} \\ \underline{\omega} \end{bmatrix}}$$

General screw motion with $\underline{\text{NONZERO}}$ rotation!!

transf. of rotation about $\omega$ by $\theta$ & then translation along $\omega$ by distance $h \cdot \theta$ :

$$\boxed{g = e^{\hat{\xi}\theta}}$$

"SCREW" : $\underline{\text{Axis}}$ $\omega$, $\underline{\text{Magnitude}}$ = distance you $\overset{d}{\underline{\text{translate}}}$ & angle $\overset{\theta}{\underline{\text{you move}}}$.

"Exp" : $(\underset{\in \mathbb{R}^6}{\xi}, \theta)$

$\uparrow$

$\xi$ , $\theta$

Pitch: If you rotate by $\pi$, but you translate by 1m:

$$\underline{h = \frac{1}{\pi}}$$

Translation: What is the screw??

$h$ = "Infinite" zero rotation, we still move!

$$h = \frac{d}{\theta \to 0}$$

$$\xi = \begin{bmatrix} v \\ 0 \end{bmatrix}$$ → $v$ is unit vec. in direction of transl.

$\theta$ → amount of translation!

$$g_{02} = g_{01} \cdot g_{12}$$