LOST jacobians 1026

Wednesday, October 26, 2022 5:08 PM

LOST SECTION 10/26

TODAY'S AGENDA:

1. Quick Review of Rigid Body Velocities
2. Manipulator Jacobian
3. Singularities of the Jacobian

Rigid Body Velocities:

- What are they??
  - They are TRANSFORMATIONS that help us find PHYSICAL vels!

What do they look like?

⇒ If we have a rigid transf $g \in SE(3)$

$$\hat{V}^s = \underset{\text{deriv}}{\dot{g}}\, g^{-1} \quad \Big\}\ \text{Spatial rigid body vel!}$$

$$\hat{V}^s = \begin{bmatrix} \underset{3\times3}{\hat{\omega}} & \underset{3\times1\ \text{vec}}{v} \\ 0 & 0 \end{bmatrix} \in \mathbb{R}^{4\times4}$$

How do we use it?

$\bar{q}_A$: When we spin / move our rigid body, how fast is q_A going??

$$\boxed{\dot{\bar{q}}_A = \hat{V}^s \bar{q}_A}$$

Velocity vec. in the spatial fr!! — Point in homog. coords in the SPATIAL fr!

$\hat{V}^s$ } transf. which HELPS US FIND physical vel.

$\hat{V}^s \in \mathbb{R}^{4\times4}$

"Vee map" takes us from MATRIX → VEC.

$\hat{V}^s \ \ [\hat{\omega}^s \ \ v^s]$ ... $V^s = [v^s$ ... $\omega^s]$

$$V = \begin{bmatrix} \vdots & \vdots \\ 0 & 0 \end{bmatrix} \qquad \boxed{V = \begin{bmatrix} \\ \omega^s \end{bmatrix} \in \mathbb{R}^6}$$

Now repr. the "idea" of this transf. in a vector!

⇒ When we take $\hat{V}^s$, we get our transform!

- Returns velocities in the SPATIAL frame!!
- Frame ATTACHED to the rigid body: "BODY FR." (B)

$$\boxed{\hat{V}^B = g^{-1}\dot{g}}$$

BODY VELOCITY!

g is the transf. btwn SPATIAL & BODY!

⇒ Takes in a point in the BODY frame & RETURNS the spatial vel. of that point AS SEEN from the basis vecs. of B.

q^B } Body fr. point:

$$\boxed{\dot{q}_B = \hat{V}^B \bar{q}_B}$$

spatial vel. of q_B SEEN from the body fr!

$$(\hat{V}^B)^\vee = V^B \in \mathbb{R}^6$$

Q: How do we relate V^B & V^s??

⇒ Want to TRANSFORM a rigid body vel. V^B from the BODY fr. INTO the spatial fr:

$$\hat{V}^s = \dot{g}g^{-1}$$

$$\hat{V}^B = g^{-1}\dot{g}$$

$$\boxed{\hat{V}^s = g\hat{V}^b g^{-1}}$$

} JUST using g, we go from $\hat{V}^b \rightarrow \hat{V}^s$

$$g = \begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix}$$

How do we write this transf. in terms of the VECTORS??

V^s & V^b??

$$\boxed{V^s = Ad_g V^b}$$

Ad_g: A matrix that TRANSFORMS velocities from body fr. into the spatial fr.

$$Ad_g : \mathbb{R}^6 \rightarrow \mathbb{R}^6 \quad \in \mathbb{R}^{6\times 6}$$

$$\boxed{Ad_g = \begin{bmatrix} R & \hat{p} R \\ 0 & R \end{bmatrix}} \in \mathbb{R}^{6\times 6}$$

entries ONLY depend on $g = \begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix}$

Summary:

V^S, V^B

TRANSF. that takes us from our POINTS in S frame to vel. in S fr.

Transf. from pts in B fr. to SPATIAL vel of those points AS SEEN from B!

☆ $$\boxed{\hat{V}^S = g_{SB} \hat{V}^B g_{SB}^{-1}}$$ ☆

☆ $$\boxed{V^S = Ad_g V^B}$$ ☆

6x1 vec.

$$V^S = \begin{bmatrix} v^S \\ \omega^S \end{bmatrix} \qquad V^B = \begin{bmatrix} v^B \\ \omega^B \end{bmatrix}$$

$\hat{p} R$??

$\begin{bmatrix} R & \hat{p} R \end{bmatrix}$

"Cross product / SKEW SYMM." matrices!

"hat maps of 3x1 vecs" $\left[\,0 \quad R\,\right]$

$\hat{\omega} v = \omega \times v$

$\omega \times r$

V^B

$$\begin{bmatrix} \underline{v}^S \\ \underline{\omega}^S \end{bmatrix} = \begin{bmatrix} R & \hat{\underline{p}} R \\ 0 & R \end{bmatrix} \begin{bmatrix} \underline{v}^B \\ \underline{\omega}^B \end{bmatrix}$$

lin. vel. $\rightarrow$
$$\begin{bmatrix} v^S \\ \omega^S \end{bmatrix} = \begin{bmatrix} R v^B + \hat{\underline{p}} R \omega^B \\ 0 \quad R \end{bmatrix}$$

2) MANIPULATOR JACOBIAN:

Q: How can we APPLY theory of rigid body vel. to find vels. assoc. with robot arms??

$\Rightarrow$ Once we have V^S, V^B we can get vel. of EVERY pt. on our robot!

"PD" Controller

Here's an Idea: we know we can descr. a RATE of spinning with $\dot{\theta}$

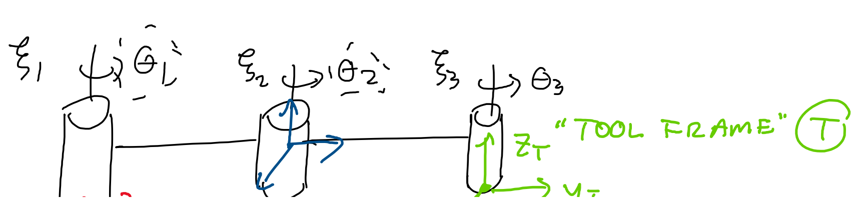
Problem: this is ONLY rotational!

What we want: SOME MAP that takes us from a VECTOR $\dot{\theta}$ of all of our joint velocities:

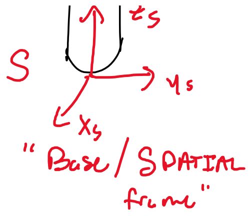
$$\dot{\theta} = \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \vdots \\ \dot{\theta}_n \end{bmatrix}$$

into the rigid body velocity of the END EFFECTOR of our robot!

Simple Robot:



$$\dot{\theta} = \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{\theta}_3 \end{bmatrix} \Rightarrow V_{ST}^S$$



Try and find: $\boxed{V_{ST}^{s}}$ "rigid body vel. btwn. spatial & tool frames!"

How do we get this transf??

=> Every axis has a twist!

$$\underline{\xi_i} = \begin{bmatrix} v_i \\ \omega_i \end{bmatrix}$$

For rev. joints, ω_i has UNIT mag.

Twists ARE rigid body vels!!

$$V^s = \begin{bmatrix} v^s \\ \omega^s \end{bmatrix}$$

let's multiply our RATE $\dot{\theta}_i$ and the twist!

$$\underline{\xi_i \dot{\theta}_i} = \begin{bmatrix} v_i \dot{\theta}_i \\ \omega_i \dot{\theta}_i \end{bmatrix}$$

"Spin about axis ω_i" w/ angular vel. $\dot{\theta}_i$

Rigid body vel. ASSOC. w/ moving joint i at rate $\dot{\theta}_i$

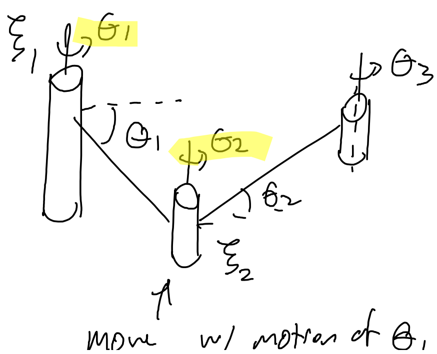
How do we find V_{st}^{s}??

Idea: Add together "components" of rigid body vel, from the motion of EACH ARM!

$$V_{st}^{s} = \xi_1 \dot{\theta}_1 + \xi_2 \dot{\theta}_2 + \xi_3 \dot{\theta}_3$$

=> This gives a CORRECT formula for V_{ST}^{S} WHEN $\theta_1 = \theta_2 = \theta_3 = 0$.

What we want: for ANY $\theta = \begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{bmatrix}$



How does this affect our rigid body velocity??

What happens to ξ_1 as we move our robot??

$\xi_1' = \xi_1$ } Doesn't

ξ_1 at NEW Config. Change:

What about ξ_2??

$$\boxed{\xi_2' = Ad_{e^{\hat{\xi}_1\theta_1}}\, \xi_2}$$

If we move with θ_1, ξ_2 "APPEARS" constant!

figured out our twist for ANY θ_1

transf. into spatial fr.

take our REFERENCE ξ_2

Shorthand: "g_1" $= e^{\hat{\xi}_1\theta_1}$

What about ξ_3??

- Now, we have TWO joints moving before it!!

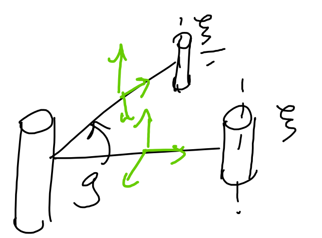
$\Rightarrow$ Must mult. by $Ad_{g_1 g_2}$ $\qquad g_2 = e^{\hat{\xi}_2\theta_2}$

$$\boxed{\xi_3' = Ad_{g_1 g_2}\, \xi_3} \qquad Ad_{g_1 g_2} = Ad_{g_1} Ad_{g_2}$$

Let's APPLY this formula:

$$\hat{V}^S = g\, \hat{V}^B g^{-1} \qquad \text{"Similarity transf"}$$

$$\dot{\bar{q}}_A = \hat{V}^S \bar{q}_A = g\, \hat{V}^B g^{-1} \bar{q}_A$$
$$= g\, \hat{V}^B \bar{q}_B$$
$$= g\, \dot{\bar{q}}_B$$
$$= \dot{\bar{q}}_A$$



$$\xi' = Ad_g\, \xi$$

Now: ADD UP velocities from each joint! All vels. in the spatial fr!

$$V^S_{ST} = \xi_1\dot{\theta}_1 + \xi_2'\dot{\theta}_2 + \xi_3'\dot{\theta}_3$$

This is NOW true for ANY θ_1, θ_2, θ_3

Goal: we wanted a MAP from $\dot{\theta} = (\dot{\theta}_1, \dot{\theta}_2, \dot{\theta}_3)$ to our rigid body vel. of the end effector

$$V_{ST}^{S} = \begin{bmatrix} | & | & | \\ \xi_1' & \xi_2' & \xi_3' \\ | & | & | \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{\theta}_3 \end{bmatrix}$$

$\theta = [\theta_1, \theta_2, \theta_3]$

$J_{st}^{S}(\theta)$ } MANIPULATOR JACOBIAN!

$$\boxed{V_{St}^{S} = J_{St}^{S}(\theta)\,\dot{\theta}}$$

returns Spatial rigid body vel!

S is for "Spatial"

$\Rightarrow$ Columns: $\begin{bmatrix} \xi_1 & \xi_2' & \xi_3' \end{bmatrix}$

"components" of velocity!

$\Rightarrow$ tell us WHERE in space our robot can move!

– Actual vel $\rightarrow$ determined by SCALING + ADDING these cols.

BODY JACOBIAN:

$$V_{ST}^{S} = \left(J_{ST}^{S}(\theta)\right)\dot{\theta}$$

V_{St}^{S}: JUST a transf!

$\Rightarrow$ J_{St}^{S} $\Rightarrow$ Gives us our SPATIAL rigid body vel!

$$\dot{q}_A = \hat{V}_{St}^{S}\, \bar{q}_A$$

☆ To be continued...

How do we get V_{St}^{B}? Maybe we can find a BODY Jacobian!

$$V_{St}^{S} = Ad_{g_{St}} V_{ST}^{B}$$

Mult. by inverse of Ad.

$$g_{ST}^{-1} = \begin{bmatrix} R^T & -R^T p \\ 0 & 1 \end{bmatrix}$$

$$Ad_{g_{St}^{-1}} V_{St}^{S} = V_{St}^{B}$$

$$Ad_{g_{ST}^{-1}} : \Big[\, R^T \quad (-R^T p)^{\wedge} R^T$$

$$A\,d \quad J^{S} \dot{\theta} \quad V^{B}$$

$$Ad_{g_{st}^{-1}} J_{st}(\theta)\,\dot{\theta} = V_{st}$$

$$\begin{bmatrix} 0 & a^T \end{bmatrix}$$

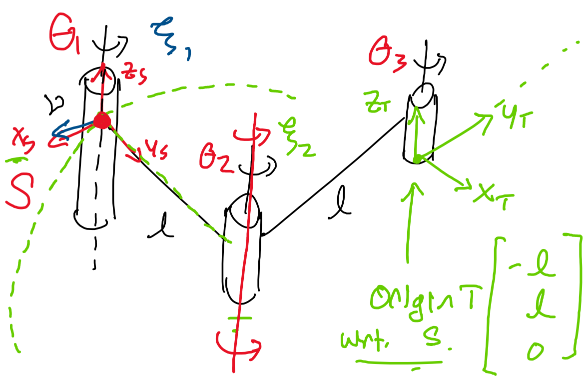
$$\underbrace{\left(Ad_{g_{st}^{-1}} J^{s}_{st}(\theta)\right)\dot{\theta}}_{\text{map from } \dot{\theta}} = V^{B}_{st}$$ } gives back our BODY vel!

$$\boxed{J^{B}_{st}(\theta) = Ad_{g_{st}^{-1}} J^{s}_{st}(\theta)}$$

"BODY JACOBIAN"

☆ $$\boxed{V^{B}_{st} = J^{B}_{st}(\theta)\,\dot{\theta}}$$ ☆ $V^{s} = J(\theta)\,\dot{\theta}$

Example: Find the spatial Jacobian $J^{s}_{st}(\theta)$ for the given config:



$$\begin{bmatrix} 0 \\ \cos\theta \\ \sin\theta \\ \vdots \end{bmatrix}$$

$$J^{s}_{st} = \begin{bmatrix} \xi_1' & \xi_2' & \xi_3' \end{bmatrix}$$

$$\theta = [0,\ \pi/2,\ 0]$$

Find our twists:

$$\xi_1 = \begin{bmatrix} -\omega_1 \times q_1 \\ \omega_1 \end{bmatrix} \qquad \omega_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \qquad -\omega_1 \times q_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\xi_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

What about ξ_2'??

$$\xi_2 = \begin{bmatrix} -\omega_2 \times q_2 \\ \omega_2 \end{bmatrix}$$

- Could use the method
- Could also use circle method!
  - Twist = vel. w/ $\dot{\theta} = 1$

$$\underline{\xi} = \begin{bmatrix} \underline{v} \\ \omega \end{bmatrix}$$

for a rev. joint, we want $\|\omega\| = 1$
$\Rightarrow \dot{\theta} = 1$ rad/s

$$V^s = \begin{bmatrix} v \\ \underline{\omega} \end{bmatrix}$$

$\|\omega\| \in \mathbb{R}$
$\Rightarrow \dot{\theta} = \|\omega\|$

"Circle Method"

$\Rightarrow$ Finding twists is hard!

- Is there an easy way to find the components of $\xi = \begin{bmatrix} v \\ \underline{\omega} \end{bmatrix}$??
- ω = Axis we rotate around, $\|\omega\| = \dot{\theta}$
- ν = VELOCITY OF A PT. MOVING THROUGH origin of S as we spin our joint!!

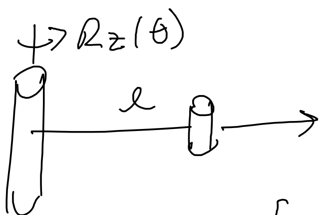
- Distance l to the axis.
- Vel. points in x dir.

$\|v\|$?? "$v = r\omega$"
$\|v\| = l$

$\dot{\theta} = \|\omega\|$ for a twist, $\|\omega\| = 1$

$$\underline{\xi_2'} = \boxed{\underline{\xi_2} = \begin{bmatrix} l \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}}$$

T.B. Ctd!



$$\xi = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\xi_2' = \begin{bmatrix} R_z(\theta) & \hat{p} R_z(\theta) \\ 0 & R_z(\theta) \end{bmatrix} \begin{bmatrix} \vdots \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

3) SINGULARITIES:

$$J_{st}^{s}(\theta) = \begin{bmatrix} | & | & | \\ \xi_1 & \xi_2' & \xi_3' \\ | & | & | \end{bmatrix}$$

⇒ Axes when added together & mult. by some $\dot{\theta}_s$ will give us a rigid body vel. for our end effector.

$$V_{st}^{s} = \dot{\theta}_1 \xi_1 + \dot{\theta}_2 \xi_2' + \dot{\theta}_3 \xi_3'$$

⇒ ξ's determine velocity directions!!

What does it mean for the Jacobian to "drop rank"

$$\alpha \xi_1 + \beta \xi_2' + \gamma \xi_3' = 0 \quad \Big\} \text{ Cols. are DEPENDENT!}$$

α, β, γ not all zero

⇒ If we have this dependence, ≥ONE of our twists is REDUNDANT!

⇒ These twists DETERMINE where our robot can go!

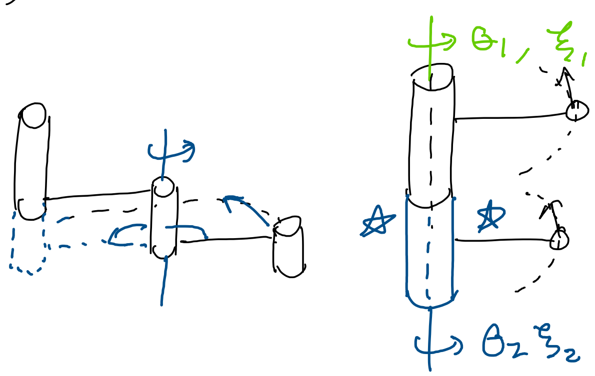
Instead of 3 indp. directions, we have 2. $\begin{bmatrix} \xi_1 & \xi_2' & \cancel{\xi_3'} \end{bmatrix}$ ex: ξ_3' could point along ξ_2'

⇒ like we LOSE a DOF in our robot!!

- Whenever we lose rank, we say J_{st}^{s} has a "singularity"

PHYSICAL EXAMPLES:

1) Two COLLINEAR REVOLUTE JOINTS:

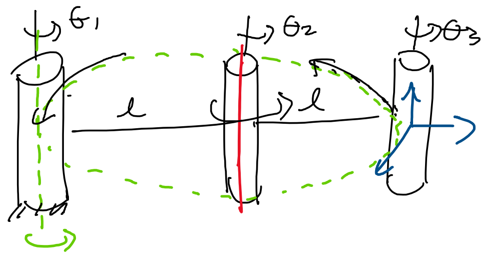


If $\dot{\theta}_1 = \dot{\theta}_2$, then BOTH arms give the SAME vel/mo:

⇒ Is just like we can move in 1 dir!!

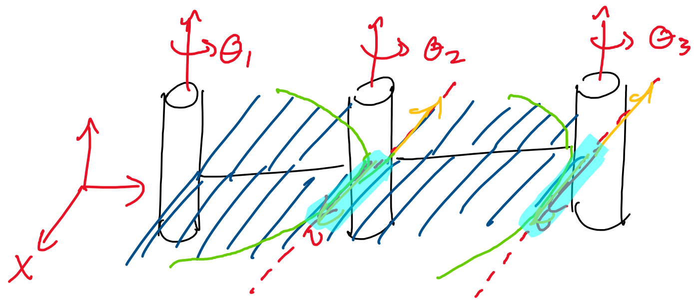
- Potential for loss of DOF!

In practice:



By moving about ξ_2, the axes align!

Ex.2: THREE PARALLEL COPLANAR REV. JOINTS:

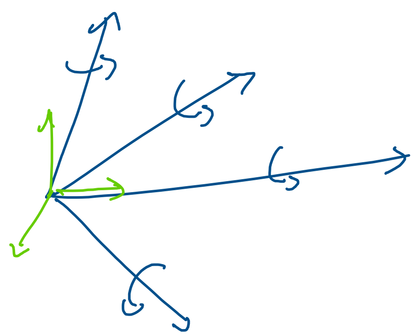


⇒ ALL of the vels. will lie in a 2D plane!

⇒ We go down to 1 DOF in velocity!!

- All vels point along X axis.

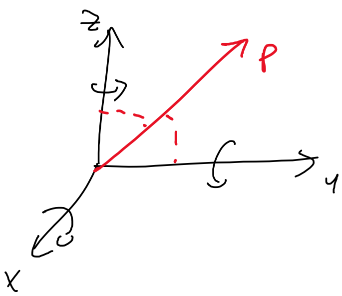
3rd Case: 4 revolute joints w/ intersecting axes:



⇒ Why we have a singularity!

- If we have 4, & only 3 components to descr. ω

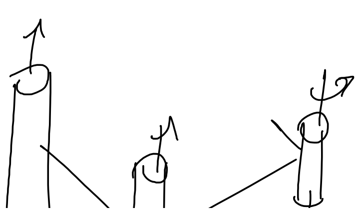
$$\xi = \begin{bmatrix} 0 \\ \omega \end{bmatrix}$$

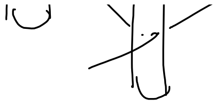


$$\omega = \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix}$$

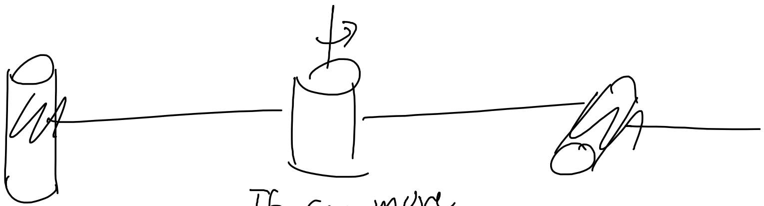
$$J = \begin{bmatrix} 0 & 0 & 0 & 0 & \cdots \\ \underline{\omega_1} & \underline{\omega_2} & \underline{\omega_3} & \underline{\omega_4} & \end{bmatrix}$$

⇒ 4 lin. indp. ωs in 3d sp.





rotational dof.



If can move
Independently of
All others by

1 DOF