Lost 921

Wednesday, September 21, 2022 5:09 PM

LOST SECTION 9/21

Agenda:

1. Types of robot joints
2. Joint Space
3. Forward kinematics!

Types of Robot Joints:

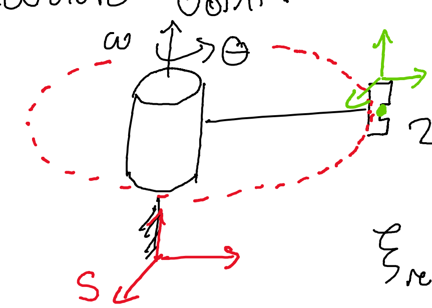
- 3 basic types of robot joints:
  1. Revolute joint
  2. Prismatic joint
  3. Screw joint!
- Can repr. ANY rigid body motion using a twist! (ξ) (xi)

$$\xi = \begin{bmatrix} v \\ \omega \end{bmatrix} \in \mathbb{R}^6$$

$$g = e^{\hat{\xi}\theta} \in \underline{SE(3)}$$

rigid body transf.

(1) Revolute Joint:



- Purely rotational motion

→ "end effector" of the arm

$$\xi_{rev} = \begin{bmatrix} -\omega \times q \\ \omega \end{bmatrix}$$

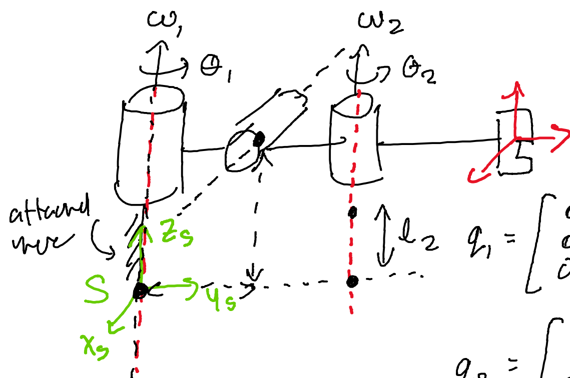
→ component of velocity that results from rotation

axis of rotation, $\|\omega\| = 1$

$$\underline{e^{\hat{\xi}_{rev}\theta}} = g$$

⇒ transf. of ROTATING about axis ω by angle θ.

- If we repr. $\xi_{rev.}$ in the SPATIAL frame, this rotates pts/vecs in the spatial frame



$$q_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\underline{q_2} = \begin{bmatrix} 0 \\ l_1 \\ 0 \end{bmatrix}$$

$$q_2 = \begin{bmatrix} 0 \\ l_1 \\ l_2 \end{bmatrix}$$

ξ_1, ξ_2 } IN frame S!

$$\xi_1 = \begin{bmatrix} -\omega_1 \times \underline{q_1} \\ \omega_1 \end{bmatrix} \to \text{in spatial fr!}$$

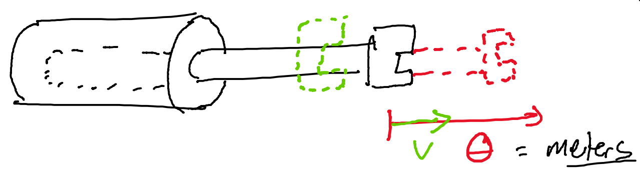
$$\xi_2 = \begin{bmatrix} -\omega_2 \times \underline{q_2} \\ \hline \omega_2 \end{bmatrix}$$

=> ω is a VECTOR!
=> Just a direction in space!

=> q is a POINT!

② Prismatic Joint

- Purely translational joint!



- Prismatic = PURELY translational joint!

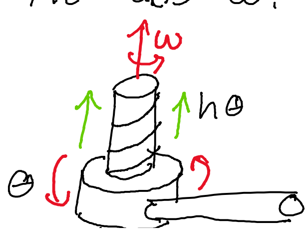
What is the twist?

$$\xi_{transl} = \begin{bmatrix} v \\ 0 \end{bmatrix} \begin{matrix} \to \text{unit vector pointing in the dir. of transl.} \\ \to \text{NO ROTATION OCCURS} \end{matrix}$$

$$g = e^{\hat{\xi}_{transl.}\theta} \Big\} \text{translation of DISTANCE } \theta \text{ in direc. } v\text{!}$$

③ SCREW JOINT

=> Screw motion: rotation about axis ω followed by a TRANSL. along the axis ω.



- Ratio of TRANSL. to Rotation is called PITCH!

$$h = \frac{d}{\theta}$$

$h = \frac{d}{\theta}$

$h \cdot \theta = \frac{d}{\theta} \cdot \theta = d$

translation from the rotational motion

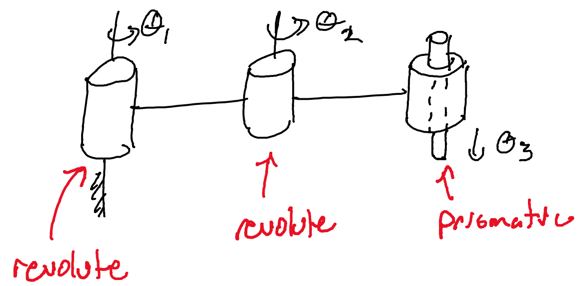
$$\xi_{screw} = \begin{bmatrix} -\omega \times q + h\omega \\ \omega \end{bmatrix} = \xi_{rev} + \xi_{prism.}$$

translation along axis ω

unit vec!

$$e^{\hat{\xi}_{screw}\theta} = g \in SE(3)$$

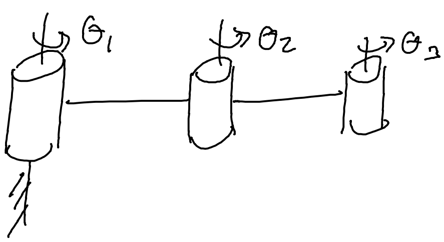
- Rotation by θ along axis ω
- Translation by $h\theta$ along axis ω



} know the transf. for each!

JOINT SPACE:

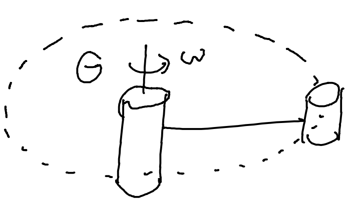
- Figure out SOME WAY to write the set of ALL possible robot configs!



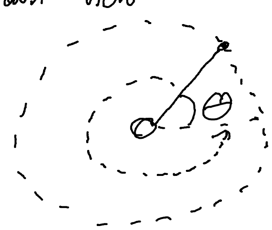
How do we describe the set of ALL combinations of joint angles?

- Set of ALL possible joint positions for our basic joints!

Revolute:



top down view



What do we know abt. θ?

1. $\theta \in \mathbb{R}$
2. After $\theta = 2\pi$, we start REPEATING joint positions!

$\theta \in [0, 2\pi)$

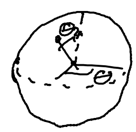
$\theta \in [0, 2\pi)$

include exclude

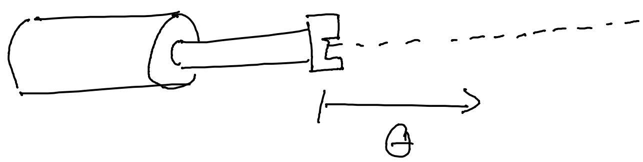
Special name for the set of revolute joint pos: $\boxed{S^1}$

Single — 1 angle to descr. pos.

S^2: descr. a Sphere!

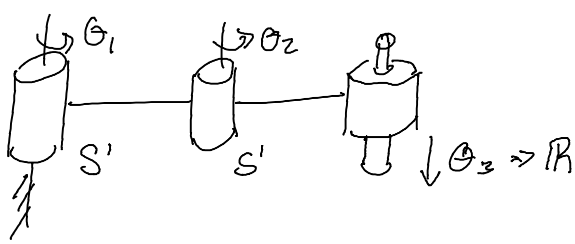


2. Prismatic:



- What restr. do we have?
  1. $\theta \in \mathbb{R}$ → As specific as we can get!

⇒ No restr. on the value!



Question: What is the set of ALL combinations of joint angles??

Call the set of ALL joint combos "Q"

X: "Cartesian product"

$$Q = S^1 \times S^1 \times \mathbb{R}$$

JOINT SPACE

give me ALL of the combinations of these sets!

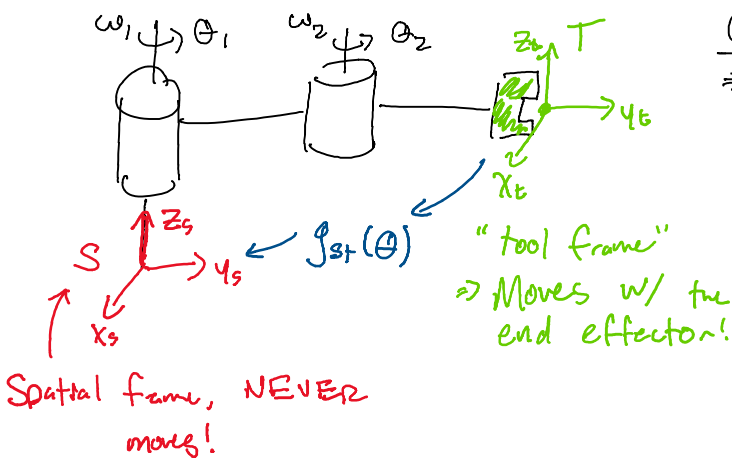
X: applied to SETS: combos
X: applied to vecs: Cross prod.

$$Q \Rightarrow \{\theta_1\} \times \{\theta_2\} \times \{\theta_3\}$$

$$\theta \in Q, \quad \theta = \begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{bmatrix}$$

⇒ write it in a vec for ease of use!
- each vector is just One combo of joint positions.

Forward Kinematics:



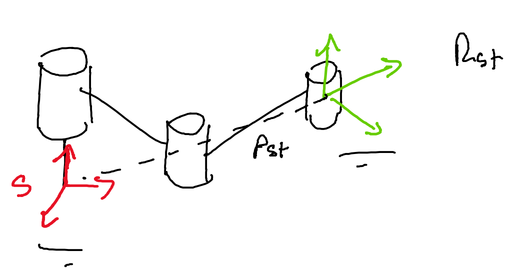
Our problem:

⇒ figure out where the hand is (end effector) in terms of the joint angles!

- How to keep track of pos & orient. of tool frame WRT the spatial frame?
- each link is rigid!
  - link motion = rigid body transf.
- Compose rigid motions!

$g_{st}(\theta)$: rigid body transf. in SE(3)

⇒ Take a point in frame T, transform it to fr. S.



} Changes w/ θ_1 & θ_2

- Form $g_{st}(\theta)$ using twists!

⇒ we know the twists for EVERY simple robot joint!
- Simplifies COMPLEX transf. into finding twists!

$g_{st}(\theta): Q \longrightarrow SE(3)$

Joint space → takes in; gives back rigid body transf. btwn. S & T.

$\theta = \begin{bmatrix} \theta_1 \\ \vdots \\ \theta_n \end{bmatrix}$

Q = set of joint angles

$\theta \in Q$

$\underset{\theta_1}{S^1} \times \underset{\theta_2}{S^1} \times \underset{\theta_3}{\mathbb{R}}$

$\begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{bmatrix}$

$SO(3)$ → JUST a rotation matrix!

↑ 3x3 matrices $R^T R = I$, $\det(R) = 1$

$SE(3)$ → set of all rigid body transf!

↑ 4x4 matrices!

$se(3)$: $\hat{\xi}$

$\xi\theta$

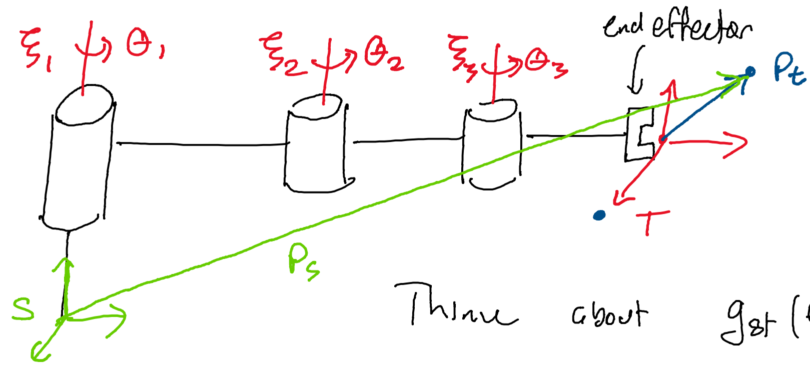
$\begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix}$, $R \in SO(3)$, $p \in \mathbb{R}^3$

$$\hat{\xi} = \begin{bmatrix} \hat{\omega} & v \\ 0 & 0 \end{bmatrix} \in \mathbb{R}^{4\times4} \text{ \} 16 elements!}$$

$$\xi = \begin{bmatrix} v \\ \omega \end{bmatrix} \in \mathbb{R}^6$$

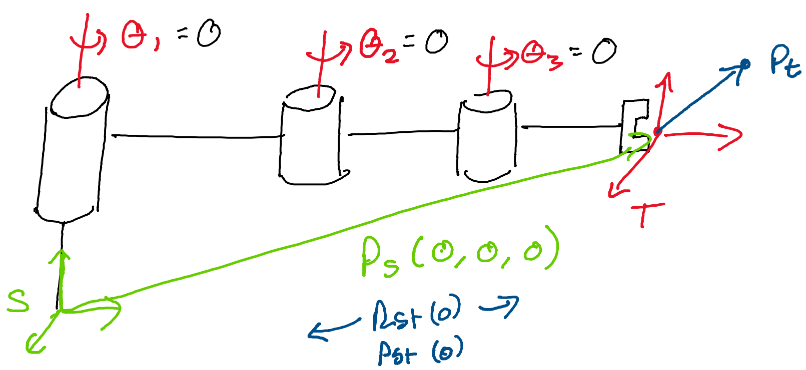
How we can use twists for robot kinematics:

$$g_{st}(\theta) := ?$$



Think about $g_{st}(\theta)$:

$$\underbrace{P_s}_{\text{that pt. in the spatial frame}} = g_{st}(\theta) \underbrace{P_t}_{\text{tool fr. point}}$$ } should be true for ANY joint angles



"Start out in REFERENCE CONFIG"

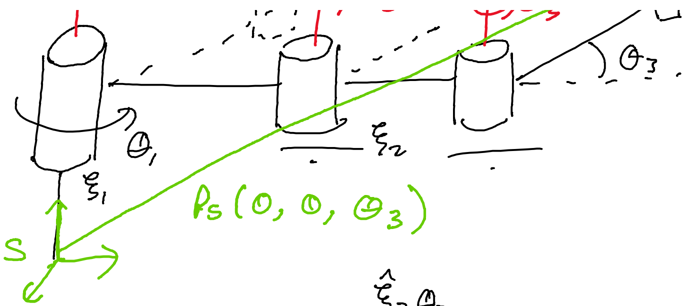
⇒ Start out by transf. P_t into the spatial frame!

$g_{st}(0)$ = Transf. between S & T in the reference config!

$$g_{st}(0)\, P_t = P_s(0,0,0)$$

⇒ have it in the spatial fr, BUT all joint angles = 0.

know that we can



find a twist ξ_3 corr. to joint 3!

$\Rightarrow$ ξ_3 is in the SPATIAL frame!

$e^{\hat{\xi}_3\theta_3}$: ROTATES in frame S about ω_3 by angle θ_3.

$$P_S(0,0,\theta_3) = e^{\hat{\xi}_3\theta_3}\, \underline{g_{st}(0)\,P_T}$$

P_T in the spatial frame in the ref. config! $P_S(0,0,0)$

Why go from outside in??

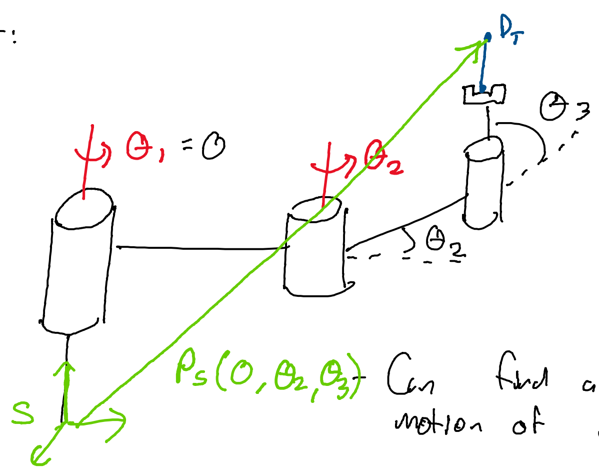
- If we start w/ ξ_1, θ_1

$$e^{\hat{\xi}_1\theta_1} \cdots \left(e^{\hat{\xi}_{n-1}\theta_{n-1}}\left(e^{\hat{\xi}_n\theta_n}\, p\right)\right)$$

$A \cdot B \cdot C$

- Outside in: Why?
  - $\Rightarrow$ Start rotating from farthest out joints, those rotations don't affect joints farther in.
  - Twists are then just the same as the ref. config.

Next Joint:



- When we move by θ_3, θ_2 joint is STILL in its original pos.
- Can use the ORIGINAL joint 2 twist ξ_2.

$P_S(0,\theta_2,\theta_3)$ Can find a twist ξ_2 for the motion of joint 2!

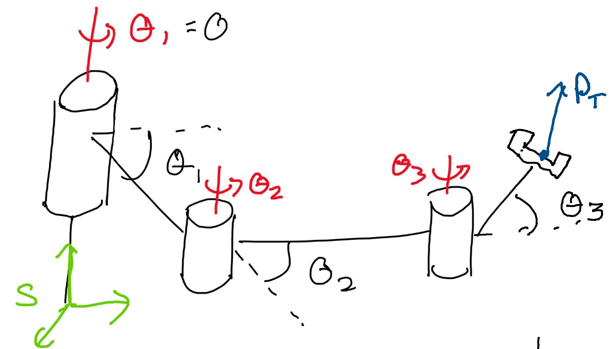
- Transf. of moving joint 2: $\underline{e^{\hat{\xi}_2\theta_2}}$

$$P_S(0,\theta_2,\theta_3) = \underbrace{e^{\hat{\xi}_2\theta_2}}_{\text{Apply rotation about joint 2}}\ \underbrace{e^{\hat{\xi}_3\theta_3}}_{\text{rotate joint 3 rotating the point}}\ \underbrace{g_{st}(0)\,P_t}_{}$$

↑ taking P_t, convert to spatial frame in REF. config.

In spatial fr.

1st Joint:



transf. about first joint:
⇒ find a twist ξ_1
→ $e^{\hat{\xi}_1 \theta_1}$ } moving by θ_1 around joint 1.

$$p_s(\theta_1, \theta_2, \theta_3) = e^{\hat{\xi}_1 \theta_1} e^{\hat{\xi}_2 \theta_2} e^{\hat{\xi}_3 \theta_3} g_{st}(0) p_t$$

⇒ have MAPPED from an arb. pt. p_t in the tool fr. to its repr. in the SPATIAL fr. after moving by θ_1, θ_2, θ_3

- Found the map we were looking for!

$$p_s(\theta_1, \theta_2, \theta_3) = g_{st}(\theta) p_t$$

$$\boxed{g_{st}(\theta) = e^{\hat{\xi}_1 \theta_1} e^{\hat{\xi}_2 \theta_2} e^{\hat{\xi}_3 \theta_3} g_{st}(0)}$$

"Product of exponentials"

Rotational Case:

$$\xi = \begin{bmatrix} v \\ \omega \end{bmatrix}$$

$$e^{\hat{\xi}\theta} = \begin{bmatrix} e^{\hat{\omega}\theta} & (I - e^{\hat{\omega}\theta})\hat{\omega}v - \omega\omega^T v\theta \\ 0 & 1 \end{bmatrix} \in \mathbb{R}^{4\times4}$$

rotation

transl. $\in \mathbb{R}^3$

3x3 rotation matrix

$$= \begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix}$$

For a twist ξ on angle θ
⇓
What is the transf g?

Prismatic case:

$$\xi = \begin{bmatrix} v \\ 0 \end{bmatrix}$$

don't rotate at all (identity)

translate by θ in direc. v

$$e^{\hat{\xi}\theta} = \begin{bmatrix} I_3 & v\theta \\ 0 & 1 \end{bmatrix}$$

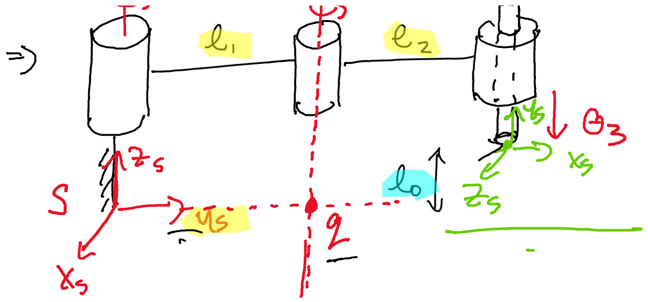
Don't compute by hand!

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \times \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix}$$

$$= (v_1\hat{i} + v_2\hat{j} + v_3\hat{k}) \times (w_1\hat{i} + w_2\hat{j} + w_3\hat{k})$$

Example:

Forward kin:



① Lay out product of exp:

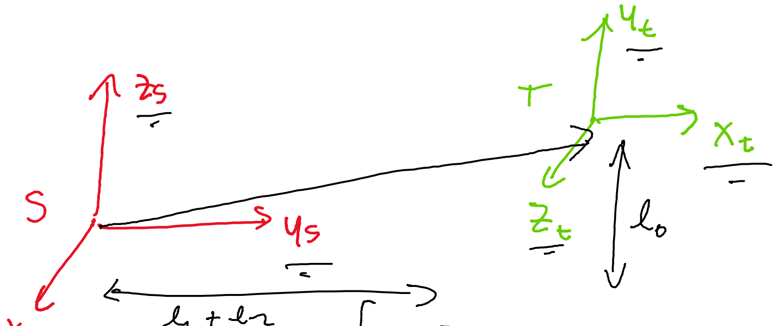
$$g_{st}(\theta) = e^{\hat{\xi}_1\theta_1} e^{\hat{\xi}_2\theta_2} e^{\hat{\xi}_3\theta_3} g_{st}(0)$$

ξ_1, ξ_2, ξ_3 ??

$g_{st}(0)$??

Solving for $g_{st}(0)$

$$g_{st}(0) = \begin{bmatrix} R_{st}(0) & p_{st}(0) \\ 0 & 1 \end{bmatrix} \in SE(3)$$



* $R_{st} \neq I$ *

$$R_{st} = \begin{bmatrix} [x_t]_s & [y_t]_s & [z_t]_s \end{bmatrix}$$

$$R_{st}(0) = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$p_{st}(0) = \begin{bmatrix} 0 \\ l_1 + l_2 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ l_0 \end{bmatrix} = \begin{bmatrix} 0 \\ l_1 + l_2 \\ l_0 \end{bmatrix}$$

$$g_{st}(0) = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & l_1 + l_2 \\ 0 & 1 & 0 & l_0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

② Solve for the twists!

$$\xi_2 = \begin{bmatrix} -\omega \times q \\ \omega \end{bmatrix} \qquad \omega = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \qquad q = \begin{bmatrix} 0 \\ l_1 \\ 0 \end{bmatrix}$$

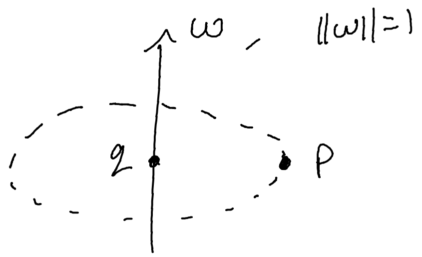
$$-\omega \times q = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \times \begin{bmatrix} 0 \\ l_1 \end{bmatrix} = \begin{bmatrix} l_1 \\ 0 \end{bmatrix}$$

$\begin{bmatrix} -1 \end{bmatrix} \quad \begin{bmatrix} 0 \end{bmatrix} \quad \begin{bmatrix} 0 \end{bmatrix}$

$$\boxed{\xi_2 := \begin{bmatrix} l_1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}}$$

Prismatic Joint: ZERO rotation: $\omega = 0$

$$\xi \stackrel{?}{=} \begin{bmatrix} -\cancel{\omega \times q} + \cancel{h\omega} \\ \cancel{\omega} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad X$$



$$\dot{p} = \omega \times (p - q)$$
$$\dot{p} = \omega \times p - \omega \times q$$
$$\dot{p} = \underset{\sim}{\hat{\omega} p} - \omega \times q$$
$$\begin{bmatrix} \dot{p} \\ 0 \end{bmatrix} = \underbrace{\begin{bmatrix} \hat{\omega} & -\overset{v}{\underline{\omega \times q}} \\ 0 & 0 \end{bmatrix}}_{\hat{\xi}} \begin{bmatrix} p \\ 1 \end{bmatrix}$$
$$\xi = \begin{bmatrix} -\omega \times q \\ \omega \end{bmatrix}$$

$$e^{\hat{\omega}_1 \theta_1} \cdot e^{\hat{\omega}_2 \theta_2}$$

<u>IF</u> $\hat{\omega}_1 \hat{\omega}_2 = \hat{\omega}_2 \hat{\omega}_1$, <u>Then</u> they commute!

$$e^{A+B} = \underline{e^A \cdot e^B} \quad \underline{IF} \quad \underline{AB = BA}$$

$$e^{A+B} = e^{B+A} = e^B \cdot e^A$$
$$\underline{e^A \cdot e^B = e^B \cdot e^A}$$

Associativity: $\left(e^{\hat{\xi}_1 \theta_1} \cdot e^{\hat{\xi}_2 \theta_2}\right) \cdot e^{\hat{\xi}_3 \theta_3}$

$((1 \quad 2) \quad 3)$

$e^1 \; e^2 \; e^3 \; e^4$

4 3 2 1

$((3\ 2)\ 1)$ C C C C