LOST 1102

Wednesday, November 2, 2022 5:10 PM

LOST SECTION 11/02: DYNAMICS

Today's Agenda:

1. Describing Energies / Why do we need dynamics
2. Motivating Lagrange
3. Examples!

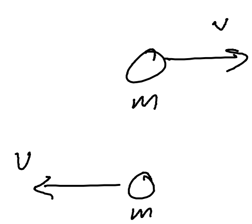
ENERGY:

- TWO MAIN TYPES of Energy: KINETIC & POTENTIAL

Kinetic Energy: "How HARD it is to STOP a moving body!"

⇒ larger mass = harder to stop → Higher KE

- larger vel = Higher KE



Just as hard to stop in ONE dir. as it is in the other!

- Gen idea: KE JUST depends on MAG. of vel!

$$\boxed{T = \frac{1}{2} m\, v \cdot v}$$

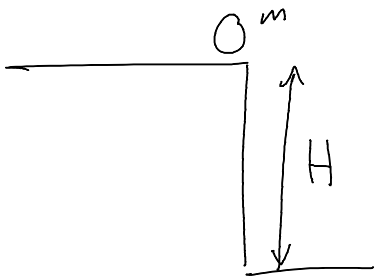
T ↑ kinetic energy; m ↑ mass of particle

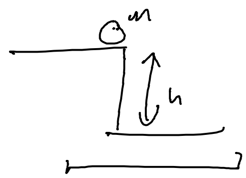
→ $v \cdot v = \|v\|^2$

⇒ A SCALAR that tells us energy assoc. w/ MOVEMENT!

POTENTIAL ENERGY:

- P.E tells us about our ABILITY to gain a high KE.





GRAVITATIONAL P.E.
"grav. const. g"
- Drop any particle
→ will accel. w/ g

⇒ HIGH height H
- b/c we have the POTENTIAL to gain lots of KE
⇓
we have a HIGH P.E.

⇒ less K.E. that we can gain!

Common val: $g = 9.81 \ m/s^2$

tells us how quickly we pick up speed!

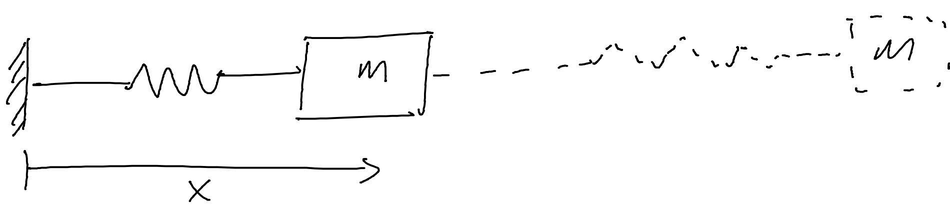
- What is important to P.E.?
  1. A height relative to a surface! (h)
  2. Mass of our particle might matter! (m)
  3. Gravitational accel (g)

$$\boxed{V_{grav} = mgh}$$

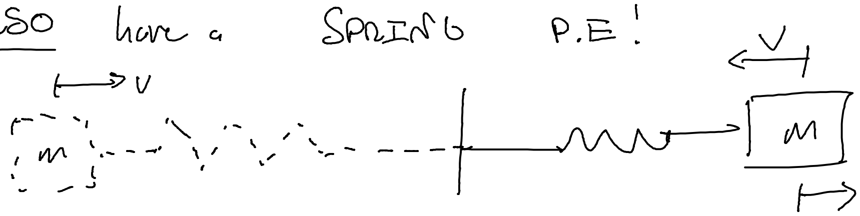
mass accel height!

ALSO a scalar!

Spring Potential Energy:



- When I let go, we gain LOTS of KE!
- ALSO have a SPRING P.E!



⇒ If we pull it in EITHER dir, we get the SAME KE "back out"

– JUST depends on HOW MUCH we stretch the spring!

$$\boxed{V_{spring} = \frac{1}{2} k x^2}$$

Spring const! (how hard it is to stretch spring!)

+ or - doesn't matter

SUMMARY:

T = "KE" ⇒ Energy of MOTION

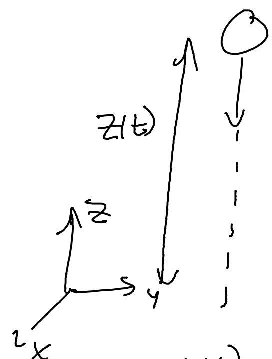
V = "PE" ⇒ how quickly we have the ABILITY to go!
"stored energy"

Care about CHANGE!

Q: What is Dynamics?

- Basic idea: find EQNS. of motion for ANY sys!

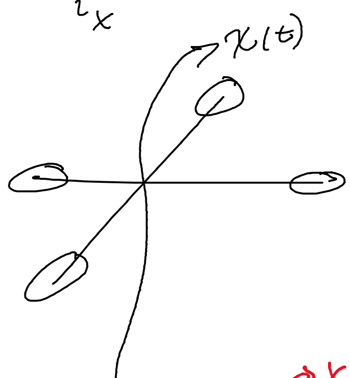
ex:



↓ gravity

Q: What is the fn. $z(t)$?

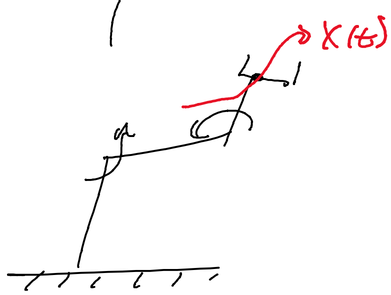
Drone:



⇒ What effect will a THRUST force have on the drone position??

- ACTUALLY FIGURE OUT $x(t) \in \mathbb{R}^3$

Robot Arms:



- How do JOINT TORQUES impact position?

How do we FIND these eqns??

$$F = ma = m\frac{d^2x}{dt^2}$$ } ACTUALLY a diff. eqn!!

F ↖ Sum of all forces!

m ↑ mass of sys

a ↖ accel. of sys!

$a = \frac{d}{dt}(v) = \frac{d}{dt}\left(\frac{dx}{dt}\right)$

$a = \frac{d^2x}{dt^2}$

⇒ How do we SOLVE

$F = m\frac{d^2x}{dt^2}$ for $x(t)$??

} 2 steps:
1. Find/set up eqn.
2. Solve/study eqn.

NOT actually the original form that Newton wrote!

$$F = \frac{dG}{dt}$$

"G" ⇒ linear momentum

$G = mv$ } vec. in $\mathbb{R}^3$ — it is a vector!

↑ mass · vel. vector!

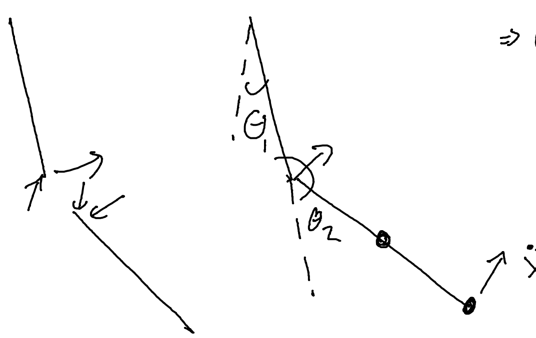
⇒ Sum of all forces = R.O.C. of MOMENTUM!

R.O.C = rate of change!

LAGRANGE'S EQUATIONS:

- What's wrong w/ $F = ma$??

⇒ Can ALSO be hard to find forces!



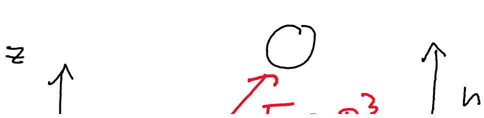
} Super hard to find $\ddot{x}$!! it's a VECTOR!

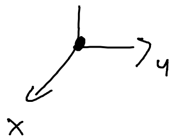
- LAGRANGE: "ABSTRACT AWAY" behind ENERGY!
  - Don't deal w/ contact forces!
  - Can IGNORE connec. / extra forces that aren't important!
  - Produces a SET of diff. eqns. IDENTICAL to $F=ma$

PROVE that Lagrange & $F=ma$ are the SAME!

↓ Gravity in the z dir!

How can we get





$F \in \mathbb{R}^3 = \begin{bmatrix} f_x \\ f_y \\ f_z \end{bmatrix}$ $\downarrow$ $X(t) = \begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix}$

$F = m\ddot{X}$

$F_x = m\ddot{x}$

$F_y = m\ddot{y}$

$F_z = m\ddot{z}$

① PICK A GEN. COORD.

⇒ Help us SPLIT UP prob. into FUNDAMENTAL directions!

- Our particle can move in x, y, & z
  - They are the SMALLEST set of coords. we need to COMPLETELY descr. motion!
- JUST measures of some positions?

② APPLY LAGRANGE:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = T \quad \Big\} \text{ "Calc. of Variations"}$$

q = ANY g.c.

Any EXTERNAL force we apply in the DIR. of q.

Take ONE DERIV. at a time of "L"

- L is called the "LAGRANGIAN"

$$L = T - V \quad \Big\} \text{ SCALAR quantity!}$$

KE of sys. P.E. of the sys.

Find Energy:

$T = \frac{1}{2} m v \cdot v$ $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ $v = \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix}$

$T = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$

$V = mgh = mgz$

$$L = T - V = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - mgz \quad \Big\} \text{ Lagrangian!}$$

TAKE OUR 1st Deriv + Interpret:

$\frac{\partial L}{\partial \dot{q}}$ let's try $q = z$ PHILOSOPHY = treat ALL but $\dot{z}$ as CONST!

$$\frac{\partial L}{\partial \dot{z}} = m\dot{z}$$

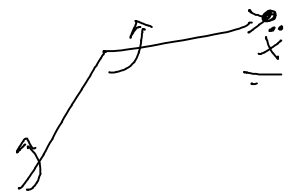
$\frac{\partial L}{\partial \dot{z}}$ is the MOMENTUM in the direc. of our g.c.

When we do $\frac{\partial L}{\partial \dot{z}}$ what is it?

$$\underset{\substack{\uparrow \\ \text{lin. momentum}}}{G} = m\begin{bmatrix}\dot{x}\\ \dot{y}\\ \dot{z}\end{bmatrix} = mv = \begin{bmatrix} m\dot{x}\\ m\dot{y}\\ m\dot{z}\end{bmatrix}$$

Summary: $\frac{\partial L}{\partial \dot{q}}$ ⇒ COMPONENT of MOMENTUM ALONG q.

TAKE $\boxed{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right)}$ ⇒ Just gives us ma!



$$\frac{d}{dt}\left(m\dot{z}\right) = m\ddot{z} \qquad a = \begin{bmatrix}\ddot{x}\\ \ddot{y}\\ \ddot{z}\end{bmatrix}$$

Just get "ma" term of $F=ma$ in the z direction!

$$\underbrace{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right)}_{ma} - \frac{\partial L}{\partial q} = \Gamma$$

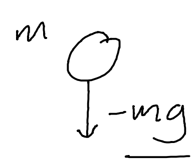
$$q = \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} \qquad \frac{\partial k}{\partial q} = \begin{bmatrix} \frac{\partial k_1}{\partial q_1} & \frac{\partial k_2}{\partial q_1} \\ \frac{\partial k_1}{\partial q_2} & \frac{\partial k_2}{\partial q_2} \end{bmatrix}$$

Now, let's take $\frac{\partial L}{\partial q}$

$$L = T - V = \cancel{\frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)} - mgz$$

I'll do the z dir. as an example.

$$\frac{\partial L}{\partial z} = -mg$$



"Zero = z down"
"O.W. = z up"

$\frac{\partial L}{\partial z}$ } Get the FORCE associated with POTENTIAL ENERGY!

"Energetic / CONSERVATIVE forces"

"Conservative forces are the grad. of a "POTENTIAL field""

$$U \Rightarrow \text{Force due to } U = -\nabla U = -\sum \frac{\partial U}{\partial x_i} x_i$$

1st

coord.

$U = -mgz \qquad \nabla U = -\frac{\partial U}{\partial z}\hat{z} = -mg\,\hat{z}$

$$\underbrace{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right)}_{\text{mass}\cdot\text{accel}} - \underbrace{\frac{\partial L}{\partial q}}_{\text{energetic force}} = \underbrace{\Gamma}_{\text{"external" forces!}}$$

$$ma - F_{consv} = \Gamma$$

$$ma = \underline{F_{consv}} + \Gamma = F$$

Γ = ALL NON CONSV. forces in dir. of our gen. coord.

ALL of our NON CONSV. forces in the z-direction!

ex: friction, air resistance

$$m\ddot{z} + mg = F_z$$

} gives a DIFF EQN. for $\underline{z(t)}$.

RIGID BODY DYNAMICS:

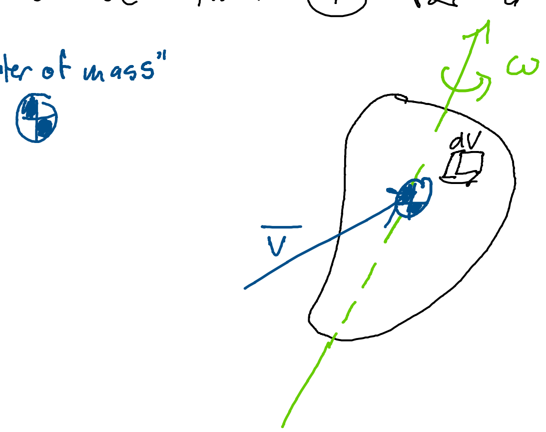
Q: How to REPEAT for rigid bodies??

- LAGRANGE STILL HOLDS for a rigid body!

$$\underline{L = T - V}$$

- How do we find T for a rigid body? How about V?

"Center of mass"



- In gen, EVERY point on the rigid body has a DIFF velocity!!

$T_{particle} = \frac{1}{2} m\, v \cdot v$

- Rotation AND translation!

Defn: KE for a rigid body is computed:

"VOLUME" element!

$$T = \frac{1}{2}\int_V \rho\, v \cdot v \, dV$$

DENSITY

vel. of ANY pt. in rigid body

$\rho V = m$

Similar STRUCTURE to $\frac{1}{2} m\, v \cdot v$

- How do we AVOID taking this integral??
- Idea: SPLIT UP into TWO PARTS!
  1. Translational (T_t)
  2. Rotational (T_r)

Velocity: JUST a vector!

How can we use $\bar{v}$??

$$v = \bar{v} + \omega \times r$$

vector FROM c.o.m. to that point in your rigid body!

Try and plug into integral:

$$T = \frac{1}{2}\int_V \rho\, v \cdot v \, dV$$

$$= \frac{1}{2}\int_V \rho \left(\bar{v} + \omega \times r\right) \cdot \left(\bar{v} + \omega \times r\right) dV$$

Volume integral over body!

$$= \frac{1}{2}\int_V \rho\, \bar{v}\cdot\bar{v} + \rho\, \text{Term2} \; dV$$

$$= \frac{1}{2}\int_V \rho\, \bar{v}\cdot\bar{v} \, dV + \frac{1}{2}\int_V \rho\, \text{Term2} \, dV$$

JUST A CONST!!

$$T = \frac{1}{2}\, \bar{v}\cdot\bar{v} \int_V \rho \, dV + \frac{1}{2}\int_V \rho\, \text{Term2} \, dV$$

Term2: $\rho(\omega\times r)\cdot(\omega\times r) + 2\bar{v}\cdot(\omega\times r)\, dV$ — "QUADRATIC FORM"

$$T = \frac{1}{2} m\, \bar{v}\cdot\bar{v} + \frac{1}{2}\int_V \rho\, \text{Term2} \, dV$$

$$\boxed{T = \underbrace{\frac{1}{2} m\, \bar{v}\cdot\bar{v}}_{T_t} + \underbrace{\frac{1}{2}\omega^T I_{com}\, \omega}_{T_r}}$$

ENTIRE KE of a rigid body!

T_t: transl. KE

T_r: ROT. K.E.

What is T_r??

I_{com}..

Shape: mult. by $\omega \in \mathbb{R}^3$ on one side!
mult. by ω^T on the other side!

$$\begin{bmatrix} \cdot & \cdot & \cdot \end{bmatrix} \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix} \begin{bmatrix} \cdot \\ \cdot \\ \cdot \end{bmatrix}$$

$I_{com} \rightarrow$ MUST be a $\underline{3\times3}$ matrix!

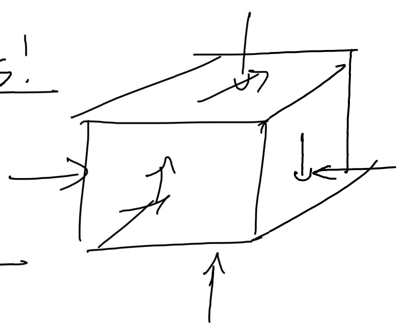
"INERTIA TENSOR" $\Rightarrow$ Can REWRITE in terms of kron. prod! $\otimes$

$$I_{xx}(x \otimes x) + I_{yy}(y \otimes y) + \ldots$$

First order tensor $\rightarrow$ 3x1 vec
2nd order $\rightarrow$ 3x3 matrix

Not in class!

Me C85
Me 108
Me T85



Stress tensor $\rightarrow$ transf. from strain into stress force

$$\begin{bmatrix} & \\ & \end{bmatrix}$$

What is the Inertia Matrix?

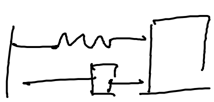
"pos. def"
all eigs > 0

$$\underline{I_{com}} = \begin{bmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{xy} & I_{yy} & I_{yz} \\ I_{xz} & I_{yz} & I_{zz} \end{bmatrix}$$

rot. about x — rot. about y — rot. about z

$\Rightarrow$ How hard it is to ROTATE our rigid body about each axis!

$\rightarrow$ How BALANCED our obj. is WRT the axes!



$$\dot{x} = \underline{A}x$$ complex part of eigenvals (A) are res. freq.

$$T = \underline{\tfrac{1}{2} m \bar{v} \cdot \bar{v}} + \tfrac{1}{2}\omega^T I \omega$$

terms of I = "how hard it is to rotate about an axis"

"Inertia matrix" $\xrightarrow{\text{Energy}}$ I_{com}

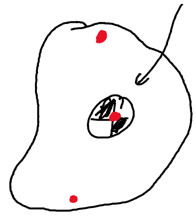
"Inertia matrix" $\xrightarrow{\text{eqns. of motion}}$ $\underline{M}$

energy "balances" around c.o.m.

Potential Energy:

$$V = mgh = \boxed{mg\bar{z}}$$

↑ Just look at C.O.M. All mass is "centered / collected" there!



↓ grav

↑ z

Q: How can we ORGANIZE our eqns. of motion??

$$M\ddot{q} + C\dot{q} + N = \Gamma$$

Matrices that DEPEND on our gen. coords!

q = vector of gen. coords! $q = \begin{bmatrix} x \\ y \\ \theta \\ \vdots \end{bmatrix}$

$$M\ddot{q} + C\dot{q} + N = \Gamma$$

M: "inertial mass matrix"

⇒ looks like ma for transl.

⇒ looks like $I\ddot{\theta}$ for rotation

- gives info about m, I

N: Vector of ALL conservative forces!!

$$\frac{\partial L}{\partial q}$$

$$N = \begin{bmatrix} \frac{\partial L}{\partial x} \\ \frac{\partial L}{\partial y} \\ \vdots \end{bmatrix}$$

"Gravity vector"

Γ → EXTERNAL / APPLIED forces! don't depend on an "energy field"

Γ is called "generalized" force vector!

⇒ Angular equiv. of force is TORQUE!

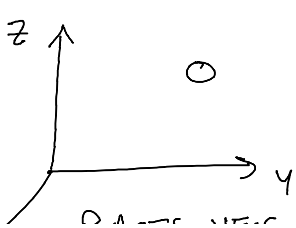
- Γ could have forces AND torques.

CORIOLIS TERM??

$$M\ddot{q} + \underline{C\dot{q}} + N = \Gamma \Rightarrow$$

$$M\ddot{q} = F$$

⇒ looks like $F = ma$!! Why do we need $C\dot{q}$??



x, y, z gen. coords.

$$p = x\hat{x} + y\hat{y} + z\hat{z}$$

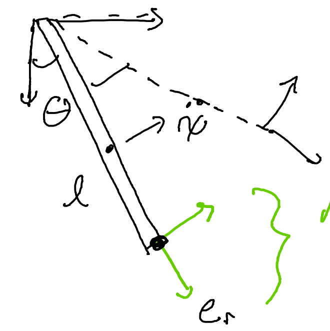
X

BASIS VECS. are CONST!!

↑ constant

$\ddot{p} = \ddot{x}\hat{x} + \ddot{y}\hat{y} + \ddot{z}\hat{z}$

Accel. for F=ma JUST depends on $\ddot{q}$ terms!!



MOVING BASIS! like polar coords!

How do we expr. ACCEL. in terms of θ??

$p = l e_r$

$\dot{p} = \dot{l} e_r + l \dot{e}_r$

$\ddot{p} = \ddot{l} e_r + \dot{l}\dot{e}_r + \dot{l}\dot{e}_r + l \ddot{e}_r$

$\Rightarrow \dot{e}_r, \ddot{e}_r \Big\}$ depend on $\theta, \dot{\theta}$

- B/C of changing coord sys, "a" in F=ma becomes MESSY & doesn't just depend on $\ddot{\theta}$. (also on $\theta, \dot{\theta}$)

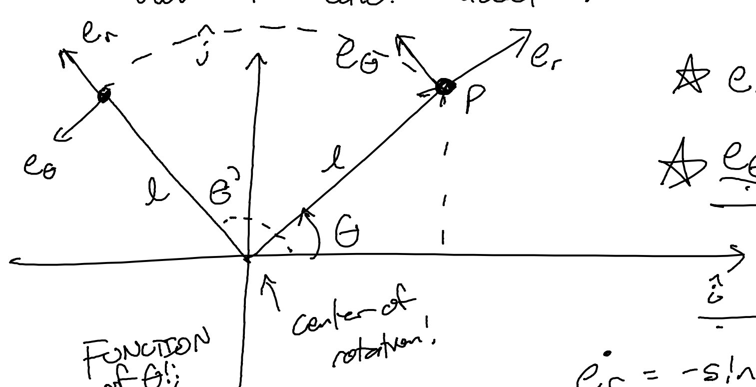
$C\dot{q} \Big\}$ "Collects" all $\theta, \dot{\theta}$ terms that contribute to accel.

$M\ddot{q} + C\dot{q}$ — depend on $q, \dot{q}$

$M\ddot{\theta} \Big\}$ Not enough!

Polar Coords: how to calc. accel??

Polar coords: (r, θ)



FUNCTION of θ!!

★ $e_r = \cos\theta\,\hat{i} + \sin\theta\,\hat{j}$ ★

★ $e_\theta = -\sin\theta\,\hat{i} + \cos\theta\,\hat{j}$ ★

$\dot{e}_r = -\sin\theta\cdot\dot{\theta}\,\hat{i} + \cos\theta\cdot\dot{\theta}\,\hat{j}$

$\dot{e}_r = \dot{\theta} e_\theta$

★ $p = l e_r$ ★ $= x\hat{i} + y\hat{j}$

$\dot{p} = \dot{l} e_r + l\dot{e}_r = \dot{l} e_r + l\dot{\theta} e_\theta$

$\ddot{p} = \ddot{l} e_r + \dot{l}\dot{e}_r + \dot{l}\dot{\theta} e_\theta + l\ddot{\theta} e_\theta + l\dot{\theta}\dot{e}_\theta$

Can show $\dot{e}_\theta = -\dot{\theta} e_r$

$\ddot{p} = \ddot{l}e_r + l\ddot{\theta}e_\theta + l\dot{\theta}e_\theta + \dot{l}\dot{\theta}e_\theta + \ldots$ plus some others (see above!)

depend on θ as well!! <u>doesn't</u> just depend on $\ddot{\theta}$

$\Rightarrow$ for $F = ma = \underline{m\ddot{p}} = m\ddot{p}\underline{(l, \dot{l}, \theta, \dot{\theta}, \ddot{\theta})}$

XM $\ddot{\theta}$