#### Last Time

### **Chapter 2 Rigid Body Motion**

- **2** Rotational motion in  $\mathbb{R}^3$ 
  - Quaternions
- 3 Rigid Motion in  $\mathbb{R}^3$ 
  - SE(3)
    - As a Configuration Space
      - Homogeneous Representation
    - SE(3) is a Group
    - SE(3) is a Rigid Body Transformation

## Recap

#### Chapter 2 Rigid Body Motion

- **3** Rigid Motion in  $\mathbb{R}^3$ 
  - Exponential coordinates of SE(3)
    - Twists
      - se(3)
      - The Exponential Map
    - Screw Motion
      - What is a Screw
        - Twist associated with a Screw
      - Screw associated with a Twist

- Forward kinematics
  - Joint Space

#### Exponential coordinates of SE(3):

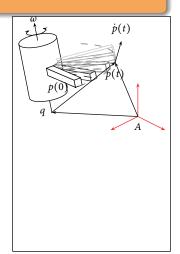
For rotational motion:

$$\begin{aligned}
\dot{p}(t) &= \omega \times (p(t) - q) \\
\begin{bmatrix} \dot{p} \\ 0 \end{bmatrix} &= \begin{bmatrix} \hat{\omega} & -\omega \times q \\ 0 & 0 \end{bmatrix} \begin{bmatrix} p \\ 1 \end{bmatrix} \\
\text{or } \dot{\overline{p}} &= \hat{\xi} \cdot \overline{p} \Rightarrow \overline{p}(t) = e^{\hat{\xi}t} \overline{p}(0)
\end{aligned}$$

where 
$$e^{\hat{\xi}t} = I + \hat{\xi}t + \frac{(\hat{\xi}t)^2}{2!} + \cdots$$

Rigid Motion

in ℝ<sup>3</sup>



#### **Exponential coordinates of** SE(3):

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Rigid Body Transformations

Rotational motion in  $\mathbb{R}$ 

Rigid Motion in  $\mathbb{R}^3$ 

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Wrenches and Reciprocal Screws

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For rotational motion:  

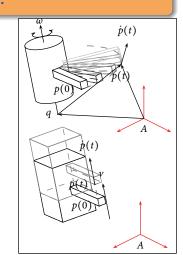
$$\dot{p}(t) = \omega \times (p(t) - q)$$

$$\begin{bmatrix} \dot{p} \\ 0 \end{bmatrix} = \begin{bmatrix} \hat{\omega} & -\omega \times q \\ 0 & 0 \end{bmatrix} \begin{bmatrix} p \\ 1 \end{bmatrix}$$
or 
$$\dot{\overline{p}} = \hat{\xi} \cdot \overline{p} \Rightarrow \overline{p}(t) = e^{\hat{\xi}t}\overline{p}(0)$$
where 
$$e^{\hat{\xi}t} = I + \hat{\xi}t + \frac{(\hat{\xi}t)^2}{2!} + \cdots$$
For translational motion:  

$$\dot{p}(t) = v$$

$$\begin{bmatrix} \dot{p}(t) \\ 0 \end{bmatrix} = \begin{bmatrix} 0 & v \\ 0 & 0 \end{bmatrix} \begin{bmatrix} p \\ 1 \end{bmatrix}$$

 $\dot{\overline{p}}(t) = \hat{\xi} \cdot \overline{p}(t) \Rightarrow \overline{p}(t) = e^{\hat{\xi}t} \overline{p}(0)$ 



#### Chapter 2 Rigid Body Motion

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- 1 Forward kinematics
  - Joint Space

#### Definition:

$$se(3) = \left\{ \begin{bmatrix} \hat{\omega} & v \\ 0 & 0 \end{bmatrix} \in \mathbb{R}^{4 \times 4} \middle| v, \omega \in \mathbb{R}^3 \right\}$$

is called the twist space. There exists a 1-1 correspondence between se(3) and  $\mathbb{R}^6$ , defined by  $\wedge : \mathbb{R}^6 \mapsto se(3)$ 

$$\xi \coloneqq \left[ \begin{array}{c} v \\ \omega \end{array} \right] \mapsto \hat{\xi} = \left[ \begin{array}{cc} \hat{\omega} & v \\ 0 & 0 \end{array} \right]$$

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#### Chapter 2 Rigid Body Motion

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Rotational motion in  $\mathbb{R}^3$ 

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Referenc

**Property 6:**  $\exp : se(3) \mapsto SE(3), \hat{\xi}\theta \mapsto e^{\xi\theta}$ 

#### Definition:

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motion in R

Rigid Motion in  $\mathbb{R}^3$ 

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Reference

**Property 6:**  $\exp: se(3) \mapsto SE(3), \hat{\xi}\theta \mapsto e^{\hat{\xi}\theta}$ 

Proof:

Let 
$$\hat{\xi} = \begin{bmatrix} \hat{\omega} & \nu \\ 0 & 0 \end{bmatrix}$$

■ If  $\omega = 0$ , then  $\hat{\xi}^2 = \hat{\xi}^3 = \dots = 0$ ,  $e^{\hat{\xi}\theta} = \begin{bmatrix} I & \nu\theta \\ 0 & 1 \end{bmatrix} \in SE(3)$ 

(continues next slide)

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If  $\omega$  is not 0, assume  $\|\omega\| = 1$ .

Define:

$$g_0 = \left[ \begin{array}{cc} I & \omega \times v \\ 0 & 1 \end{array} \right], \hat{\xi}' = g_0^{-1} \cdot \hat{\xi} \cdot g_0 = \left[ \begin{array}{cc} \hat{\omega} & h\omega \\ 0 & 0 \end{array} \right]$$

where  $h = \omega^T \cdot v$ .

$$e^{\hat{\xi}\theta} = e^{g_0 \cdot \hat{\xi}' \cdot g_0^{-1}} = g_0 \cdot e^{\hat{\xi}'\theta} \cdot g_0^{-1}$$

and as

$$\hat{\xi}^{\prime 2} = \begin{bmatrix} \hat{\omega}^2 & 0 \\ 0 & 0 \end{bmatrix}, \hat{\xi}^{\prime 3} = \begin{bmatrix} \hat{\omega}^3 & 0 \\ 0 & 0 \end{bmatrix}$$

we have 
$$e^{\hat{\xi}'\theta} = \begin{bmatrix} e^{\hat{\omega}\theta} & h\omega\theta \\ 0 & 1 \end{bmatrix} \Rightarrow e^{\hat{\xi}\theta} = \begin{bmatrix} e^{\hat{\omega}\theta} & (I - e^{\hat{\omega}\theta})\hat{\omega}\nu + \omega\omega^T\nu\theta \\ 0 & 1 \end{bmatrix}$$

Chapter Rigid Body Motion

Rigid Body Transformations

Rotational motion in  $\mathbb{R}^3$ 

 $\begin{array}{c} \text{Rigid Motion} \\ \text{in } \mathbb{R}^3 \end{array}$ 

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$$p(\theta) = e^{\hat{\xi}\theta} \cdot p(0) \Rightarrow g_{ab}(\theta) = e^{\hat{\xi}\theta}$$

If there is offset,

$$g_{ab}(\theta) = e^{\hat{\xi}\theta}g_{ab}(0)$$
 (Why?)

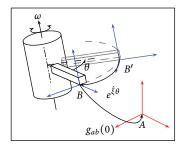


Figure 2.14

**Property 7:**  $exp : se(3) \mapsto SE(3)$  is onto.

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**Property 7:**  $exp : se(3) \mapsto SE(3)$  is onto.

Proof:

Let  $g = (p, R), R \in SO(3), p \in \mathbb{R}^3$ 

Case 1: (R = I) Let

$$\hat{\xi} = \begin{bmatrix} 0 & \frac{p}{\|p\|} \\ 0 & 0 \end{bmatrix}, \theta = \|p\| \Rightarrow e^{\hat{\xi}\theta} = g = \begin{bmatrix} I & p \\ 0 & 1 \end{bmatrix}$$

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## □ Screws, twists and screw motion:

Rigid Motion in  $\mathbb{R}^3$ 



Screw attributes

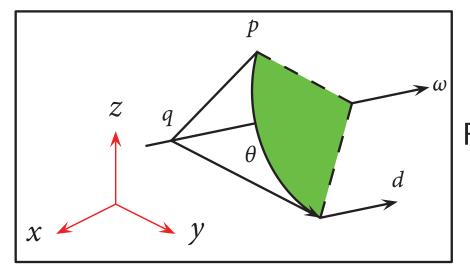


Figure 2.15

 $h = \frac{d}{\theta}(\theta = 0, h = \infty), d = h \cdot \theta$ Pitch:

Axis:  $l = \{q + \lambda \omega | \lambda \in \mathbb{R}\}$ Magnitude:  $M = \theta$ 

## □ Screws, twists and screw motion:

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Screw attributes

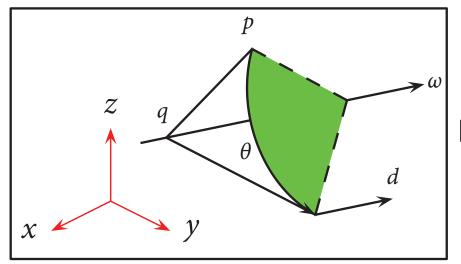


Figure 2.15

Pitch:  $h = \frac{d}{\theta}(\theta = 0, h = \infty), d = h \cdot \theta$ 

Axis:  $l = \{q + \lambda \omega | \lambda \in \mathbb{R}\}$ 

Magnitude:  $M = \theta$ 

### **Definition:**

A **screw** S consists of an axis l, pitch h, and magnitude M. A **screw motion** is a rotation by  $\theta = M$  about l, followed by translation by  $h\theta$ , parallel to l. If  $h = \infty$ , then, translation about  $\nu$  by  $\theta = M$ 

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Corresponding  $g \in SE(3)$ :

 $g \cdot p = q + e^{\hat{\omega}\theta}(p - q) + h\theta\omega$ 

$$g \cdot \left[ \begin{array}{c} p \\ 1 \end{array} \right] = \left[ \begin{array}{c} e^{\hat{\omega}\theta} \\ 0 \end{array} \right] \left( I - e^{\hat{\omega}\theta} \right) q + h\theta\omega \left[ \begin{array}{c} p \\ 1 \end{array} \right] \Rightarrow$$

$$g = \begin{bmatrix} e^{\hat{\omega}\theta} & (I - e^{\hat{\omega}\theta})q + h\theta\omega \\ 0 & 1 \end{bmatrix}$$

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Corresponding  $g \in SE(3)$ :

$$g \cdot p = q + e^{\hat{\omega}\theta}(p - q) + h\theta\omega$$

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$$g = \begin{bmatrix} e^{\hat{\omega}\theta} & (I - e^{\hat{\omega}\theta})q + h\theta\omega \end{bmatrix}$$

On the other hand...

$$e^{\hat{\xi}\theta} = \begin{bmatrix} e^{\hat{\omega}\theta} & (I - e^{\hat{\omega}\theta})\omega \times v + \omega\omega^T v\theta \\ 0 & 1 \end{bmatrix}$$

If we let  $v = -\omega \times q + h\omega$ , then

$$(I - e^{\hat{\omega}\theta})(-\hat{\omega}^2 q) = (I - e^{\hat{\omega}\theta})(-\omega\omega^T q + q) = (I - e^{\hat{\omega}\theta})q$$

Thus, 
$$e^{\hat{\xi}\theta} = g$$

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Corresponding  $g \in SE(3)$ :

$$g \cdot p = q + e^{\hat{\omega}\theta}(p - q) + h\theta\omega$$

$$g \cdot \begin{bmatrix} p \\ 1 \end{bmatrix} = \begin{bmatrix} e^{\hat{\omega}\theta} & (I - e^{\hat{\omega}\theta})q + h\theta\omega \end{bmatrix} \begin{bmatrix} p \\ 1 \end{bmatrix} \Rightarrow$$

$$g = \begin{bmatrix} e^{\hat{\omega}\theta} & (I - e^{\hat{\omega}\theta})q + h\theta\omega \end{bmatrix}$$

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On the other hand...

$$e^{\hat{\xi}\theta} = \begin{bmatrix} e^{\hat{\omega}\theta} & (I - e^{\hat{\omega}\theta})\omega \times \nu + \omega\omega^T \nu\theta \\ 0 & 1 \end{bmatrix}$$

If we let  $v = -\omega \times q + h\omega$ , then

$$(I - e^{\hat{\omega}\theta})(-\hat{\omega}^2 q) = (I - e^{\hat{\omega}\theta})(-\omega\omega^T q + q) = (I - e^{\hat{\omega}\theta})q$$

Thus, 
$$e^{\hat{\xi}\theta} = g$$

For pure rotation (h = 0):  $\xi = (-\omega \times q, \omega)$ 

For pure translation:  $g = \begin{bmatrix} I & v\theta \\ 0 & 1 \end{bmatrix}$ ,  $\Rightarrow \xi = (v, 0)$ , and  $e^{\hat{\xi}\theta} = g$ 

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## □ Screw associated with a twist:

$$\xi = (v, \omega) \in \mathbb{R}^6$$

Pitch: 
$$h = \begin{cases} \frac{\omega^T v}{\|\omega\|^2}, & \text{if } \omega \neq 0 \\ \infty, & \text{if } \omega = 0 \end{cases}$$

2 Axis: 
$$l = \begin{cases} \frac{\omega \times v}{\|\omega\|^2} + \lambda \omega, & \lambda \in \mathbb{R}, \text{ if } \omega \neq 0 \\ 0 + \lambda v, & \lambda \in \mathbb{R}, \text{ if } \omega = 0 \end{cases}$$

Magnitude: 
$$M = \begin{cases} \|\omega\|, & \text{if } \omega \neq 0 \\ \|\nu\|, & \text{if } \omega = 0 \end{cases}$$

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## □ Screw associated with a twist:

$$\xi = (\nu, \omega) \in \mathbb{R}^6$$

Pitch: 
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2 Axis: 
$$l = \begin{cases} \frac{\omega \times v}{\|\omega\|^2} + \lambda \omega, & \lambda \in \mathbb{R}, \text{ if } \omega \neq 0 \\ 0 + \lambda v, & \lambda \in \mathbb{R}, \text{ if } \omega = 0 \end{cases}$$

Magnitude: 
$$M = \begin{cases} \|\omega\|, & \text{if } \omega \neq 0 \\ \|\nu\|, & \text{if } \omega = 0 \end{cases}$$

## **Special cases:**

- h = 0, Pure rotation (revolute joint)

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Screw	Twist: $\hat{\xi} heta$
Case 1:	
Pitch: $h = \infty$	$\theta = M$ ,
Axis: $l = \{q + \lambda v     v   = 1, \lambda \in \mathbb{R}\}$	$\hat{\xi} = \begin{bmatrix} 0 & v \\ 0 & 0 \end{bmatrix}$
Magnitude: M	
Case 2:	
Pitch: $h \neq \infty$	$\theta = M$ ,
Axis: $l = \{q + \lambda \omega   \ \omega\  = 1, \lambda \in \mathbb{R}\}$	$\hat{\xi} = \begin{bmatrix} \hat{\omega} & -\hat{\omega}q + h\omega \end{bmatrix}$
Magnitude: M	$\begin{bmatrix} \zeta - [0 \\ 0 \end{bmatrix}$

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Screw	Twist: $\hat{\xi} heta$
Case 1:	
Pitch: $h = \infty$	$\theta = M$ ,
Axis: $l = \{q + \lambda v     v   = 1, \lambda \in \mathbb{R}\}$	$\hat{\xi} = \begin{bmatrix} 0 & v \\ 0 & 0 \end{bmatrix}$
Magnitude: M	ζ – [ 0 0 ]
Case 2:	
Pitch: $h \neq \infty$	$\theta = M$ ,
Axis: $l = \{q + \lambda \omega   \ \omega\  = 1, \lambda \in \mathbb{R}\}$	$\hat{\xi} = \begin{bmatrix} \hat{\omega} & -\hat{\omega}q + h\omega \\ 0 & 0 \end{bmatrix}$
Magnitude: M	ζ – [ 0

### **Definition: Screw Motion**

Rotation about an axis by  $\theta=M$ , followed by translation about the same axis by  $h\theta$ 

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## **2.3** Rigid motion in $\mathbb{R}^3$

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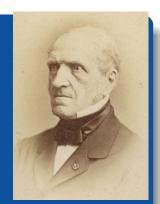
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## Theorem 2 (Chasles):

Every rigid body motion can be realized by a rotation about an axis combined with a translation parallel to that axis.



1793-1880

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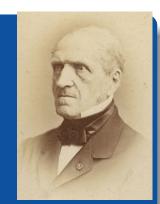
Velocity of a Rigid Body

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Reference

## Theorem 2 (Chasles):

Every rigid body motion can be realized by a rotation about an axis combined with a translation parallel to that axis.



1793-1880

### Proof:

For  $\hat{\xi} \in se(3)$ :

$$\hat{\xi} = \hat{\xi}_1 + \hat{\xi}_2 = \begin{bmatrix} \hat{\omega} & -\omega \times q \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & h\omega \\ 0 & 0 \end{bmatrix}$$
$$\begin{bmatrix} \hat{\xi}_1, \hat{\xi}_2 \end{bmatrix} = 0 \Rightarrow e^{\hat{\xi}\theta} = e^{\hat{\xi}_1\theta}e^{\hat{\xi}_2\theta}$$

† End of Section †

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# Chapter 3 Manipulator Kinematics

Forward kinematics

Inverse Kinematics

Manipulator Jacobian

Redundant Manipulators

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- 1 Forward kinematics
- 2 Inverse Kinematics
- 3 Manipulator Jacobian
- 4 Redundant Manipulators
- 5 Reference

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## 3

## 3.1 Forward kinematics

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Forward kinematics

Inverse Kinematics

Manipulator Jacobian

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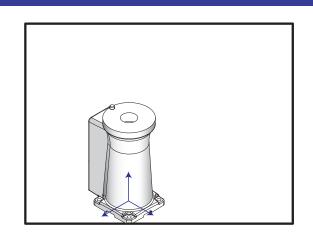
(a) Adept Cobra i600 (SCARA) Figure 3.1

Chapter 3 Manipulator Kinematics

Forward kinematics

Kinematics

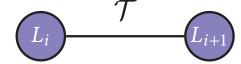




- (a) Adept Cobra i600 (SCARA)
- (b) Forward kinematics of SCARA Figure 3.1
- ♦ Lower Pair Joints:

revolute joint  $S^1 \mapsto SO(2)$  prismatic joint  $\mathbb{R} \mapsto T(1)$ 

 $\mathcal{R}$ 



♦ Forward kinematics:



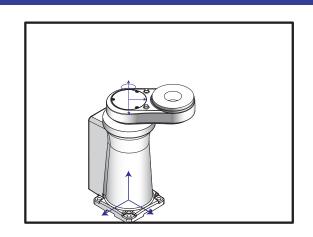
base: stationary

Chapter 3 Manipulator Kinematics

Forward kinematics

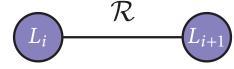
Kinematics

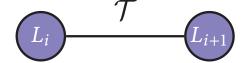




- (a) Adept Cobra i600 (SCARA)
  - (b) Forward kinematics of SCARA Figure 3.1
- ♦ Lower Pair Joints:

revolute joint  $S^1 \mapsto SO(2)$  prismatic joint  $\mathbb{R} \mapsto T(1)$ 





♦ Forward kinematics:



base: stationary

Link 1: first movable link

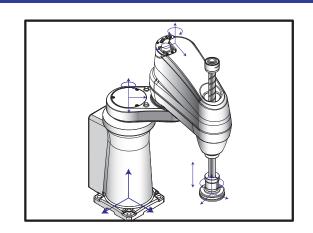


Chapter 3 Manipulator Kinematics

Forward kinematics

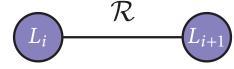
Kinematics

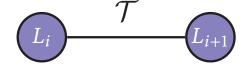




- (a) Adept Cobra i600 (SCARA)
  - (b) Forward kinematics of SCARA Figure 3.1
- ♦ Lower Pair Joints:

revolute joint  $S^1 \mapsto SO(2)$  prismatic joint  $\mathbb{R} \mapsto T(1)$ 





♦ Forward kinematics:

joint n joint 1

base: stationary Link 1: first movable link

# Robots in Practice SCARA

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Kinematics

Forward kinematics

Inverse Kinematics

Manipulator Jacobian

Redundant Manipulators

Reference

## **□** Joint space:

Revolute joint:  $S^1$ ,  $\theta_i \in S^1$  or  $\theta_i \in [-\pi, \pi]$ 

Prismatic joint:  $\mathbb{R}$ 

Joint space:  $Q: \underline{S^1 \times \cdots \times S^1} \times \underline{\mathbb{R} \times \cdots \times \mathbb{R}}$ 

no. of R joint no. of P joint

## □ Joint space:

Revolute joint:  $S^1$ ,  $\theta_i \in S^1$  or  $\theta_i \in [-\pi, \pi]$ 

Prismatic joint:  $\mathbb{R}$ 

Smatic joint:  $\mathbb{R}$ Joint space:  $Q: \underline{S^1 \times \cdots \times S^1} \times \underline{\mathbb{R} \times \cdots \times \mathbb{R}}$ 

no. of *R* joint no. of *P* joint

Chapter 3 Manipulator Kinematics

Forward kinematics

Kinematics

Adept 
$$Q: S^1 \times S^1 \times S^1 \times \mathbb{R}$$
  
Elbow  $Q = \Gamma^6: \underbrace{S^1 \times \cdots \times S^1}_{6}$ 

Chapter 3 Manipulator Kinematics

Forward kinematics

Kinematics

## □ Joint space:

Revolute joint:  $S^1, \theta_i \in S^1 \text{ or } \theta_i \in [-\pi, \pi]$ 

Prismatic joint:  $\mathbb{R}$ 

Joint space:  $Q: \underline{S^1 \times \cdots \times S^1} \times \underline{\mathbb{R} \times \cdots \times \mathbb{R}}$ no. of *R* joint no. of *P* joint

Adept 
$$Q: S^1 \times S^1 \times S^1 \times \mathbb{R}$$
  
Elbow  $Q = \Gamma^6: \underbrace{S^1 \times \cdots \times S^1}_{6}$ 

Reference (nominal) joint config:  $\theta = (0, 0, ..., 0) \in Q$ Reference (nominal) end-effector config:  $g_{st}(0) \in SE(3)$ 

#### Chapter 3 Manipulator Kinematics

Forward kinematics

Kinematics

## □ Joint space:

Revolute joint:  $S^1, \theta_i \in S^1 \text{ or } \theta_i \in [-\pi, \pi]$ 

Prismatic joint:  $\mathbb{R}$ 

Smatic joint:  $\mathbb{R}$ Joint space:  $Q: \underline{S^1 \times \cdots \times S^1} \times \underline{\mathbb{R} \times \cdots \times \mathbb{R}}$ no. of *R* joint no. of *P* joint

Adept 
$$Q: S^1 \times S^1 \times S^1 \times \mathbb{R}$$
  
Elbow  $Q = \Gamma^6: \underbrace{S^1 \times \cdots \times S^1}_{6}$ 

Reference (nominal) joint config:  $\theta = (0, 0, ..., 0) \in Q$ Reference (nominal) end-effector config:  $g_{st}(0) \in SE(3)$ 

Arbitrary configuration  $g_{st}(\theta)$ :

$$g_{st}:\theta\in Q\mapsto g_{st}(\theta)\in SE(3)$$

# **Chapter 2 Rigid Body Motion**

- 3 Rigid Motion in  $\mathbb{R}^3$ 
  - Exponential coordinates of SE(3)
    - Twists
    - *se*(3)
    - The Exponential Map
  - Screw Motion
    - What is a Screw
    - Twist associated with a Screw
    - Screw associated with a Twist

- 1 Forward kinematics
  - Joint Space

## □ Classical Approach:

 $g_{st}(\theta_1, \theta_2) = g_{st}(\theta_1) \cdot g_{l_1 l_2} \cdot g_{l_2 t}$ 

Disadvantage: A coordinate frame needed for each link

Chapter
3 Manipulator
Kinematics

Forward kinematics

Inverse Kinematics

Manipulator Jacobian

Redundant Manipulators

## □ Classical Approach:

$$g_{st}(\theta_1, \theta_2) = g_{st}(\theta_1) \cdot g_{l_1 l_2} \cdot g_{l_2 t}$$

**Disadvantage:** A coordinate frame needed for each link

□ The product of exponentials formula:

Consider Fig 3.2.

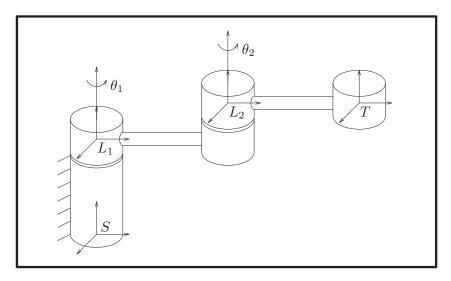


Figure 3.2: A two degree of freedom manipulator

(Continues next slide)

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3 Manipulator
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