Last Time

Chapter 2 Rigid Body Motion

2 Rotational motion in $\mathbb{R}^3$

- *Quaternions*

3 Rigid Motion in $\mathbb{R}^3$

- SE(3)
  - As a Configuration Space
  - *Homogeneous Representation*
  - *SE(3) is a Group*
  - *SE(3) is a Rigid Body Transformation*

Comparison between Rotation & Rigid-body Motion

Recap

special orthogonal

Rotation :

$R \in SO(3)$

(i) Configuration R_{ab}

(ii) change of Ref. Frame

$$q_a = R_{ab} q_b$$

$$R_{ac} = R_{ab} R_{bc}$$

$\in \mathbb{R}^3$ $\quad \in SO(3)$ $\quad$ special Euclidean

Rigid-body Motion :

$$g = (p, R) \in SE(3), \quad g = \begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix} \in SE(3)$$

(i) Configuration g_{ab}

(ii) change of Ref. Frame

$\boxed{Q}$ $q_a = g_{ab} q_b$?

$$\begin{bmatrix} q_a \\ 1 \end{bmatrix} = g_{ab} \begin{bmatrix} q_b \\ 1 \end{bmatrix}$$

$q_a = p_{ab} + R_{ab} q_b$

$$g_{ac} = g_{ab} g_{bc}$$

Today

Chapter 2 Rigid Body Motion

3 Rigid Motion in $\mathbb{R}^3$

- *Exponential coordinates of SE(3)*
  - *Twists*
  - *se(3)*
  - *The Exponential Map*
- *Screw Motion*
  - *What is a Screw*
  - *Twist associated with a Screw*
  - *Screw associated with a Twist*

Chapter 3 Manipulator Kinematics

1 Forward kinematics

- *Joint Space*

2.3 Rigid motion in $\mathbb{R}^3$

Exponential coordinates of $SE(3)$:

For rotational motion:

$$\dot{p}(t) = \omega \times (p(t) - q)$$

$$\begin{bmatrix} \dot{p} \\ 0 \end{bmatrix} = \begin{bmatrix} \hat{\omega} & -\omega \times q \\ 0 & 0 \end{bmatrix} \begin{bmatrix} p \\ 1 \end{bmatrix}$$

or $\dot{\bar{p}} = \hat{\xi} \cdot \bar{p} \Rightarrow \bar{p}(t) = e^{\hat{\xi}t}\bar{p}(0)$

where $e^{\hat{\xi}t} = I + \hat{\xi}t + \frac{(\hat{\xi}t)^2}{2!} + \cdots$

$\hat{\xi} =$

$\dot{\bar{p}} = \hat{\xi}\bar{p} \Rightarrow \bar{p}(t) = e^{\hat{\xi}t}\bar{p}(0)$

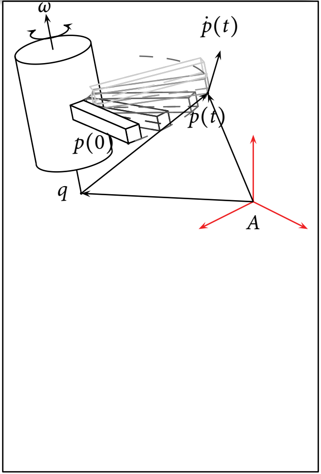

Figure 2.13

Exponential coordinates of $SE(3)$:

For rotational motion:

$$\dot{p}(t)=\omega \times (p(t)-q)$$

$$\begin{bmatrix} \dot{p} \\ 0 \end{bmatrix}=\begin{bmatrix} \hat{\omega} & -\omega \times q \\ 0 & 0 \end{bmatrix}\begin{bmatrix} p \\ 1 \end{bmatrix}$$

$$\text{or } \dot{\bar{p}}=\hat{\xi}\cdot\bar{p} \Rightarrow \bar{p}(t)=e^{\hat{\xi}t}\bar{p}(0)$$

$$\text{where } e^{\hat{\xi}t}=I+\hat{\xi}t+\frac{(\hat{\xi}t)^2}{2!}+\cdots$$

For translational motion:

$$\dot{p}(t)=v$$

$$\begin{bmatrix} \dot{p}(t) \\ 0 \end{bmatrix}=\boxed{\begin{bmatrix} 0 & v \\ 0 & 0 \end{bmatrix}}\begin{bmatrix} p \\ 1 \end{bmatrix}$$

$$\underline{\dot{\bar{p}}(t)=\hat{\xi}\cdot\bar{p}(t)} \Rightarrow \boxed{\bar{p}(t)=e^{\hat{\xi}t}\bar{p}(0)}$$

$\hat{\xi}$

$$\hat{\xi}=\begin{bmatrix} 0 & v \\ 0 & 0 \end{bmatrix}$$

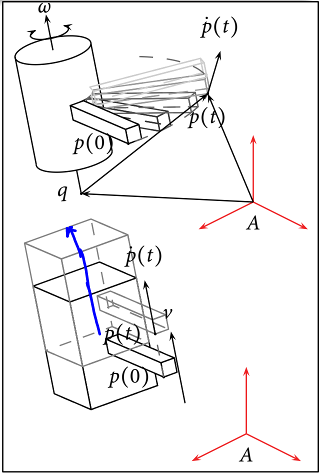

Figure 2.13

Today

Chapter 2 Rigid Body Motion

3 Rigid Motion in $\mathbb{R}^3$

- *Exponential coordinates of SE(3)*
  - *Twists*
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- *Screw Motion*
  - *What is a Screw*
  - *Twist associated with a Screw*
  - *Screw associated with a Twist*

Chapter 3 Manipulator Kinematics

1 Forward kinematics

- *Joint Space*

2.3 Rigid motion in $\mathbb{R}^3$

$\boxed{Q}$ Is $A \in se(3)$ skew symmetric? ✗

Definition:

$$se(3) = \left\{ \begin{bmatrix} \hat{\omega} & v \\ 0 & 0 \end{bmatrix} \in \mathbb{R}^{4\times 4} \,\middle|\, \underline{v}, \underline{\omega} \in \mathbb{R}^3 \right\}$$

is called the <u>twist</u> space. There exists a 1-1 correspondence between $se(3)$ and $\mathbb{R}^6$, defined by $\wedge : \mathbb{R}^6 \mapsto se(3)$

$$\underbrace{\xi}_{\in \mathbb{R}^6} := \underbrace{\begin{bmatrix} v \\ \omega \end{bmatrix}}_{6\times 1} \overset{\wedge}{\mapsto} \hat{\xi} = \underbrace{\begin{bmatrix} \hat{\omega} & v \\ 0 & 0 \end{bmatrix}}_{4\times 4}$$

For Pure Rotation

$$\xi = \begin{bmatrix} -\omega \times q \\ \omega \end{bmatrix}$$

For Pure Translation

$$\xi = \begin{bmatrix} v \\ 0 \end{bmatrix}$$

$$\xi \in \mathbb{R}^6 \underset{\vee}{\overset{\wedge}{\rightleftarrows}} \hat{\xi} \in se(3) \underset{\log}{\overset{\exp}{\rightleftarrows}} \underline{e^{\hat{\xi}} \in SE(3)}$$

$$\xi = \begin{bmatrix} v \\ \omega \end{bmatrix}, \qquad \hat{\xi} = \begin{bmatrix} \hat{\omega} & v \\ 0 & 0 \end{bmatrix}$$

Today

Chapter 2 Rigid Body Motion

3 Rigid Motion in $\mathbb{R}^3$

- *Exponential coordinates of SE(3)*
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- *Screw Motion*
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Chapter 3 Manipulator Kinematics

1 Forward kinematics

- *Joint Space*

2.3 Rigid motion in $\mathbb{R}^3$

Definition:

$$se(3) = \left\{ \begin{bmatrix} \hat{\omega} & v \\ 0 & 0 \end{bmatrix} \in \mathbb{R}^{4\times4} \middle| v, \omega \in \mathbb{R}^3 \right\}$$

is called the twist space. There exists a 1-1 correspondence between $se(3)$ and $\mathbb{R}^6$, defined by $\wedge : \mathbb{R}^6 \mapsto se(3)$

$$\xi := \begin{bmatrix} v \\ \omega \end{bmatrix} \mapsto \hat{\xi} = \begin{bmatrix} \hat{\omega} & v \\ 0 & 0 \end{bmatrix}$$

Property 6: $\exp : se(3) \mapsto SE(3), \hat{\xi}\theta \mapsto e^{\hat{\xi}\theta}$

Definition:

$$se(3) = \left\{ \begin{bmatrix} \hat{\omega} & v \\ 0 & 0 \end{bmatrix} \in \mathbb{R}^{4\times 4} \middle| v, \omega \in \mathbb{R}^3 \right\}$$

is called the twist space. There exists a 1-1 correspondence between $se(3)$ and $\mathbb{R}^6$, defined by $\wedge : \mathbb{R}^6 \mapsto se(3)$

$$\xi := \begin{bmatrix} v \\ \omega \end{bmatrix} \mapsto \hat{\xi} = \begin{bmatrix} \hat{\omega} & v \\ 0 & 0 \end{bmatrix}$$

Property 6: $\exp : se(3) \mapsto SE(3), \hat{\xi}\theta \mapsto \boxed{e^{\hat{\xi}\theta}}$ $\subset SE(3)$

Proof :

Let $\hat{\xi} = \begin{bmatrix} \hat{\omega} & v \\ 0 & 0 \end{bmatrix}$

- If $\omega = 0$, then $\hat{\xi}^2 = \hat{\xi}^3 = \cdots = 0$, $e^{\hat{\xi}\theta} = \begin{bmatrix} I & v\theta \\ 0 & 1 \end{bmatrix} \in SE(3)$

(continues next slide)

$$\hat{\xi} = \begin{bmatrix} \hat{\omega} & v \\ 0 & 0 \end{bmatrix} \qquad \text{I/ } \underline{\underline{\omega = 0}}, \quad \hat{\xi} = \begin{bmatrix} 0 & v \\ 0 & 0 \end{bmatrix}$$

$$\hat{\xi}^2 = \begin{bmatrix} 0 & v \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & v \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0$$

$$\hat{\xi}^3 = \hat{\xi}^4 = \dots = 0$$

$$e^{\hat{\xi}\theta} = I + \hat{\xi}\theta + \frac{(\hat{\xi}\theta)^2}{2!} \dots$$

$$= I + \begin{bmatrix} 0 & v\theta \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} I & v\theta \\ 0 & 1 \end{bmatrix} \in SE(3)\,? \quad \checkmark$$

2.3 Rigid motion in $\mathbb{R}^3$

- If ω is not 0, assume $\|\omega\| = 1$.

Define:

$$g_0 = \begin{bmatrix} I & \omega \times v \\ 0 & 1 \end{bmatrix}, \hat{\xi}' = g_0^{-1} \cdot \hat{\xi} \cdot g_0 = \begin{bmatrix} \hat{\omega} & h\omega \\ 0 & 0 \end{bmatrix}$$

where $h = \omega^T \cdot v$.

$$e^{\hat{\xi}\theta} = e^{g_0 \cdot \hat{\xi}' \cdot g_0^{-1}} = g_0 \cdot e^{\hat{\xi}'\theta} \cdot g_0^{-1}$$

and as

$$\hat{\xi}'^2 = \begin{bmatrix} \hat{\omega}^2 & 0 \\ 0 & 0 \end{bmatrix}, \hat{\xi}'^3 = \begin{bmatrix} \hat{\omega}^3 & 0 \\ 0 & 0 \end{bmatrix}$$

we have

$$e^{\hat{\xi}'\theta} = \begin{bmatrix} e^{\hat{\omega}\theta} & h\omega\theta \\ 0 & 1 \end{bmatrix} \Rightarrow \boxed{e^{\hat{\xi}\theta} = \begin{bmatrix} e^{\hat{\omega}\theta} & (I - e^{\hat{\omega}\theta})\hat{\omega}v + \omega\omega^T v\theta \\ 0 & 1 \end{bmatrix}}$$

$$= g_0 \, e^{\hat{\xi}'\theta} \, g_0^{-1}$$

Want to show $e^{\hat{\xi}\theta} \in SE(3)$

$$e^{\hat{\xi}\theta} = e^{I(\hat{\xi}\theta)I} = e^{\overset{g_0 \in SE(3)}{g_0 g_0^{-1}}(\hat{\xi}\theta)\, g_0 g_0^{-1}} = g_0\, e^{\underbrace{g_0^{-1}\hat{\xi} g_0}_{\hat{\xi}'}\,\theta}\, g_0^{-1}$$

$$\left(e^{PAP^{-1}} = P e^{A} P^{-1}\right)$$

step (i) $\hat{\xi}' \triangleq g_0^{-1}\hat{\xi} g_0$

step (ii) $e^{\hat{\xi}'\theta}$

step (iii) $g_0 e^{\hat{\xi}'\theta} g_0^{-1}$

---

step (i) : $g_0 = \begin{bmatrix} I & \omega\times v \\ 0 & 1 \end{bmatrix} \Longrightarrow g_0^{-1} = \begin{bmatrix} I & -\omega\times v \\ 0 & 1 \end{bmatrix}$

$$\begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} R^T & -R^T p \\ 0 & 1 \end{bmatrix}$$

$$\hat{\xi}' = \begin{bmatrix} I & -\omega\times v \\ 0 & 1 \end{bmatrix}\begin{bmatrix} \hat{\omega} & v \\ 0 & 0 \end{bmatrix}\begin{bmatrix} I & +\omega\times v \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} I & -\omega\times v \\ 0 & 1 \end{bmatrix}\begin{bmatrix} \hat{\omega} & +\hat{\omega}^2 v + v \\ 0 & 0 \end{bmatrix}$$

$$\hat{\omega}^2 = \omega\omega^T - I$$

$$= \begin{bmatrix} I & -\omega\times v \\ 0 & 1 \end{bmatrix}\begin{bmatrix} \hat{\omega} & \omega\omega^T v \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} \hat{\omega} & \omega\omega^T v \\ 0 & 0 \end{bmatrix} \in se(3)$$

Step (ii) $e^{\hat{\xi}'\theta} = I + \hat{\xi}'\theta + \frac{(\hat{\xi}'\theta)^2}{2!} + \frac{(\hat{\xi}'\theta)^3}{3!} + \cdots$

$$\hat{\xi}' = \begin{bmatrix} \hat{\omega} & \omega\omega^T v \\ 0 & 0 \end{bmatrix}$$

$$\hat{\xi}'^2 = \begin{bmatrix} \hat{\omega} & \omega\omega^T v \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \hat{\omega} & \omega\omega^T v \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} \hat{\omega}^2 & 0 \\ 0 & 0 \end{bmatrix} \qquad \underbrace{\hat{\omega}\omega}_{0}\omega^T v$$

$$\hat{\xi}'^3 = \begin{bmatrix} \hat{\omega} & \omega\omega^T v \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \hat{\omega}^2 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} \hat{\omega}^3 & 0 \\ 0 & 0 \end{bmatrix}$$

$$e^{\hat{\xi}'\theta} = \left[\begin{array}{c|c} I + \hat{\omega}\theta + \frac{(\hat{\omega}\theta)^2}{2!} + \cdots & \omega\omega^T v\theta \\ \hline 0 & 1 \end{array}\right] = \begin{bmatrix} e^{\hat{\omega}\theta} & \omega\omega^T v\theta \\ 0 & 1 \end{bmatrix}$$

$\in SE(3)$? ✓

$g_0 e^{\hat{\xi}'\theta} g_0^{-1} \in SE(3)$ □

Rotations

(iii) Transform Points / Frames

$$\dot{p} = \omega \times p$$

$$p(t) = e^{\hat{\omega} t} p(0)$$

$$p(\theta) = e^{\hat{\omega}\theta} p(0)$$

$$R(\theta) = e^{\hat{\omega}\theta} R(0)$$

Rigid body Motion

(iii) Transform Points / Frames

$$\begin{bmatrix} p(\theta) \\ 1 \end{bmatrix} = e^{\hat{\xi}\theta} \begin{bmatrix} p(0) \\ 1 \end{bmatrix}$$

$$g(\theta) = e^{\hat{\xi}\theta} g(0)$$

2.3 Rigid motion in $\mathbb{R}^3$

$$p(\theta) = e^{\hat{\xi}\theta} \cdot p(0) \Rightarrow g_{ab}(\theta) = e^{\hat{\xi}\theta}$$

If there is offset,

$$g_{ab}(\theta) = e^{\hat{\xi}\theta} g_{ab}(0)(\text{ Why?})$$

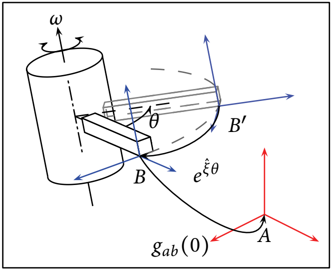

Figure 2.14

2.3 Rigid motion in $\mathbb{R}^3$

Property 7: $exp : se(3) \mapsto SE(3)$ is onto.

i.e., $\forall g \in SE(3)$, $\exists\ \xi \in \mathbb{R}^6, \theta \in \mathbb{R}$ s.t. $e^{\hat{\xi}\theta} = g$

$$g = \begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix}$$

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Chapter 3 Manipulator Kinematics

1 Forward kinematics

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2.3 Rigid motion in $\mathbb{R}^3$

□ Screws, twists and screw motion:

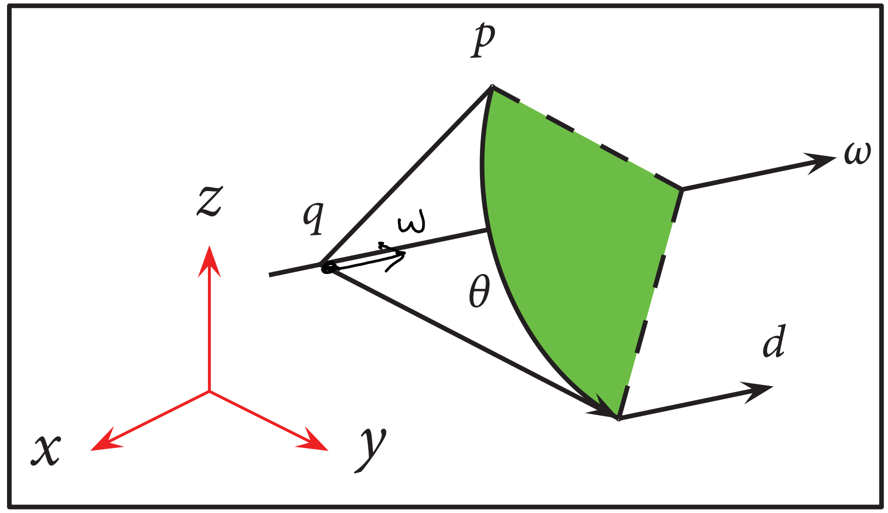


Figure 2.15

Screw attributes

(2) Pitch: $h = \frac{d}{\theta} (\theta = 0, h = \infty), d = h \cdot \theta$

(1) Axis: $l = \{q + \lambda\omega | \lambda \in \mathbb{R}\}$

Magnitude: $M = \theta$

2.3 Rigid motion in $\mathbb{R}^3$

□ Screws, twists and screw motion:

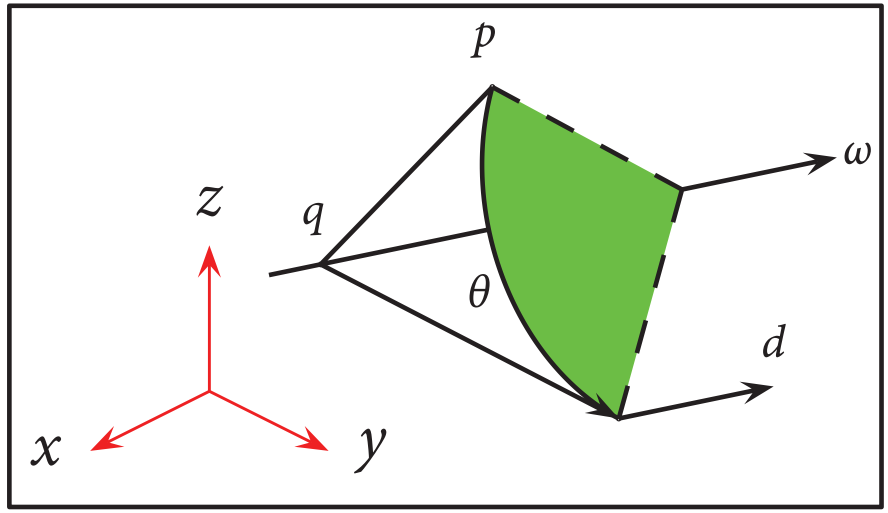


Figure 2.15

Screw attributes

Pitch: $h = \frac{d}{\theta}(\theta = 0, h = \infty), d = h \cdot \theta$

Axis: $l = \{q + \lambda\omega | \lambda \in \mathbb{R}\}$

Magnitude: $M = \theta$

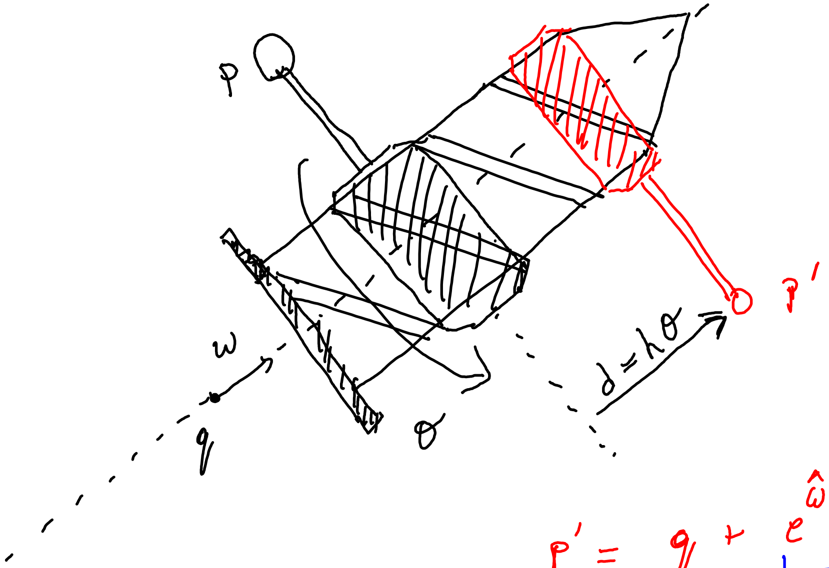
Definition:

A **screw** S consists of an axis l, pitch h, and magnitude M. A **screw motion** is a rotation by $\theta = M$ about l, followed by translation by $h\theta$, parallel to l. If $h = \infty$, then, translation about v by $\theta = M$

Screw Motion:

(1) Axis $l = \{\underline{\underline{q}} + \lambda \underline{\underline{\omega}} \mid \lambda \in \mathbb{R}\}$

$S(l, h, \theta)$



(2) Pitch $h = \dfrac{d}{\theta}$

(3) Magnitude $M = \theta$

$$p' = q + \underbrace{e^{\hat{\omega}\theta}(p-q)}_{\text{due to Rotation}} + \underbrace{h\theta\omega}_{\text{due to translation}}$$

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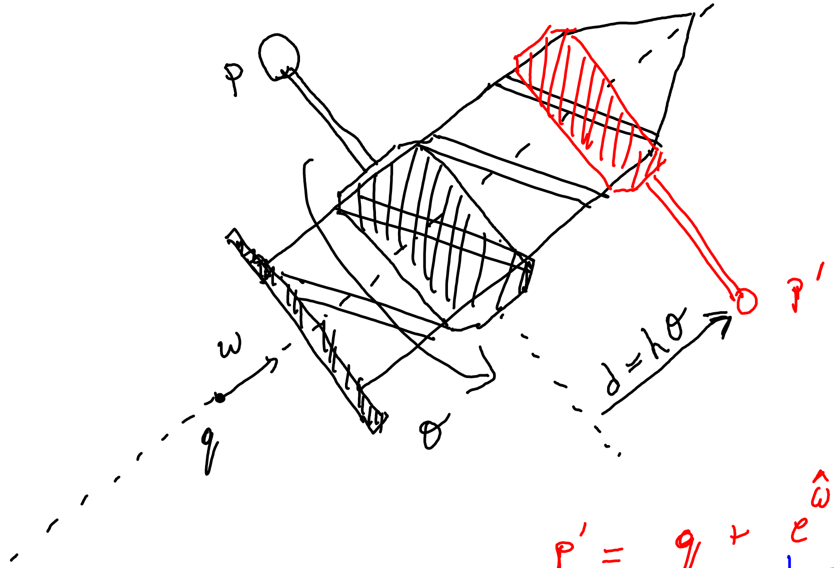
Screw Motion :



(1) Axis $l = \{ q + \lambda \omega \mid \lambda \in \mathbb{R} \}$

$S(l, h, \theta)$

(2) Pitch $h = \frac{d}{\theta}$

(3) Magnitude $M = \theta$

$$p' = q + \underbrace{e^{\hat{\omega}\theta}(p-q)}_{\text{due to Rotation}} + \underbrace{h\theta\omega}_{\text{due to translation}}$$

Find Twist for above screw :

$$\begin{bmatrix} p' \\ 1 \end{bmatrix} = \underbrace{\begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix}}_{g \in SE(3) \text{ corresponding to screw motion}} \begin{bmatrix} p \\ 1 \end{bmatrix} = \begin{bmatrix} e^{\hat{\omega}\theta} & (I - e^{\hat{\omega}\theta})q + h\theta\omega \\ 0 & 1 \end{bmatrix} \begin{bmatrix} p \\ 1 \end{bmatrix}$$

Twist $\hat{\xi} = \begin{bmatrix} \hat{\omega} & v \\ 0 & 0 \end{bmatrix}$, $\|\omega\| = 1$

$$e^{\hat{\xi}\theta} = \begin{bmatrix} e^{\hat{\omega}\theta} & (I - e^{\hat{\omega}\theta})\omega \times v + \omega\omega^T v\theta \\ 0 & 1 \end{bmatrix}$$

(From earlier)

Claim : $v = -\omega \times q + h\omega$

2.3 Rigid motion in $\mathbb{R}^3$

Corresponding $g \in SE(3)$:

$$g \cdot p = q + e^{\hat{\omega}\theta}(p - q) + h\theta\omega$$

$$g \cdot \begin{bmatrix} p \\ 1 \end{bmatrix} = \begin{bmatrix} e^{\hat{\omega}\theta} & (I - e^{\hat{\omega}\theta})q + h\theta\omega \\ 0 & 1 \end{bmatrix} \begin{bmatrix} p \\ 1 \end{bmatrix} \Rightarrow$$

$$g = \begin{bmatrix} e^{\hat{\omega}\theta} & (I - e^{\hat{\omega}\theta})q + h\theta\omega \\ 0 & 1 \end{bmatrix}$$

2.3 Rigid motion in $\mathbb{R}^3$

Corresponding $g \in SE(3)$:

$$g \cdot p = q + e^{\hat{\omega}\theta}(p - q) + h\theta\omega$$

$$g \cdot \begin{bmatrix} p \\ 1 \end{bmatrix} = \begin{bmatrix} e^{\hat{\omega}\theta} & (I - e^{\hat{\omega}\theta})q + h\theta\omega \\ 0 & 1 \end{bmatrix} \begin{bmatrix} p \\ 1 \end{bmatrix} \Rightarrow$$

$$g = \begin{bmatrix} e^{\hat{\omega}\theta} & (I - e^{\hat{\omega}\theta})q + h\theta\omega \\ 0 & 1 \end{bmatrix}$$

On the other hand...

$$e^{\hat{\xi}\theta} = \begin{bmatrix} e^{\hat{\omega}\theta} & (I - e^{\hat{\omega}\theta})\omega \times v + \omega\omega^T v\theta \\ 0 & 1 \end{bmatrix}$$

If we let $v = -\omega \times q + h\omega$, then

$$(I - e^{\hat{\omega}\theta})(-\hat{\omega}^2 q) = (I - e^{\hat{\omega}\theta})(-\omega\omega^T q + q) = (I - e^{\hat{\omega}\theta})q$$

Thus, $e^{\hat{\xi}\theta} = g$

2.3 Rigid motion in $\mathbb{R}^3$

Corresponding $g \in SE(3)$:

$$g \cdot p = q + e^{\hat{\omega}\theta}(p - q) + h\theta\omega$$

$$g \cdot \begin{bmatrix} p \\ 1 \end{bmatrix} = \begin{bmatrix} e^{\hat{\omega}\theta} & (I - e^{\hat{\omega}\theta})q + h\theta\omega \\ 0 & 1 \end{bmatrix} \begin{bmatrix} p \\ 1 \end{bmatrix} \Rightarrow$$

$$g = \begin{bmatrix} e^{\hat{\omega}\theta} & (I - e^{\hat{\omega}\theta})q + h\theta\omega \\ 0 & 1 \end{bmatrix}$$

On the other hand...

$$e^{\hat{\xi}\theta} = \begin{bmatrix} e^{\hat{\omega}\theta} & (I - e^{\hat{\omega}\theta})\omega \times v + \omega\omega^T v\theta \\ 0 & 1 \end{bmatrix}$$

If we let $v = -\omega \times q + h\omega$, then

$$(I - e^{\hat{\omega}\theta})(-\hat{\omega}^2 q) = (I - e^{\hat{\omega}\theta})(-\omega\omega^T q + q) = (I - e^{\hat{\omega}\theta})q$$

Thus, $e^{\hat{\xi}\theta} = g$

For pure rotation $(h = 0)$: $\xi = (-\omega \times q, \omega)$

For pure translation: $g = \begin{bmatrix} I & v\theta \\ 0 & 1 \end{bmatrix}$, $\Rightarrow \xi = (v, 0)$, and $e^{\hat{\xi}\theta} = g$