Last Time

Chapter 2 Rigid Body Motion

2 Rotational motion in $\mathbb{R}^3$

- The *Exponential Map*
- Rodrigues *Formula*
- *Euler Angles*

Recap

Other Parametrizations of $SO(3)$:

- XYZ fixed angles (or Roll-Pitch-Yaw angle)

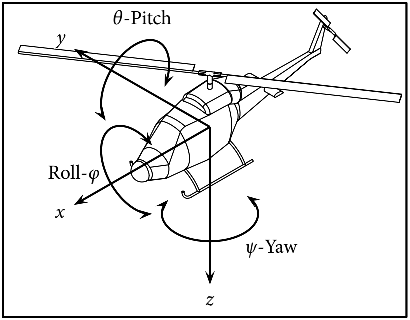

Figure 2.8

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2.2 Rotational Motion in $\mathbb{R}^3$

- XYZ fixed angles (or Roll-Pitch-Yaw angle) Continued

$$R_x(\varphi) := e^{\hat{x}\varphi} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\varphi & -\sin\varphi \\ 0 & \sin\varphi & \cos\varphi \end{bmatrix}$$

$$R_y(\theta) := e^{\hat{y}\theta} = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$$

$$R_z(\psi) := e^{\hat{z}\psi} = \begin{bmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{aligned} R_{ab} &= R_x(\varphi)R_y(\theta)R_z(\psi) \\ &= \begin{bmatrix} c_\theta c_\psi & -c_\theta s_\psi & s_\theta \\ s_\varphi s_\theta c_\psi + c_\varphi s_\psi & -s_\varphi s_\theta s_\psi + c_\varphi c_\psi & -s_\varphi c_\theta \\ -c_\varphi s_\theta c_\psi + s_\varphi s_\psi & c_\varphi s_\theta s_\psi + s_\varphi c_\psi & c_\varphi c_\theta \end{bmatrix} \end{aligned}$$

2.2 Rotational Motion in $\mathbb{R}^3$

- ZYX Euler angle

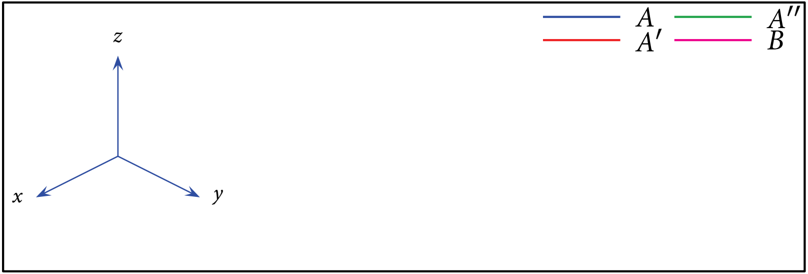

Figure 2.9

2.2 Rotational Motion in $\mathbb{R}^3$

- ZYX Euler angle

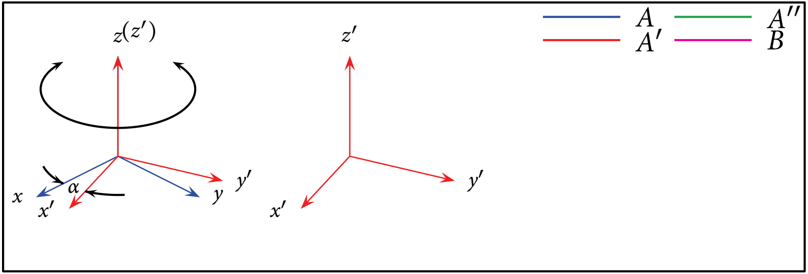

Figure 2.9

$$R_{aa'} = R_z(\alpha)$$

2.2 Rotational Motion in $\mathbb{R}^3$

- ZYX Euler angle

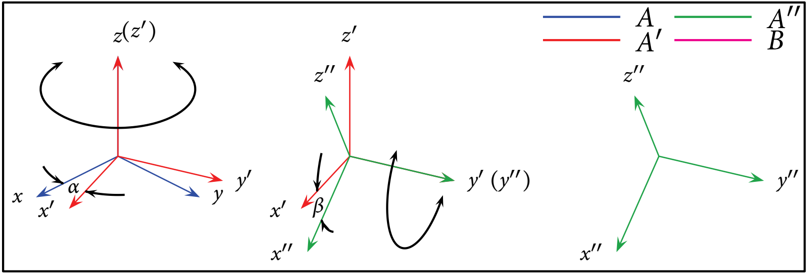

Figure 2.9

$$R_{aa'} = R_z(\alpha) \qquad R_{a'a''} = R_y(\beta)$$

2.2 Rotational Motion in $\mathbb{R}^3$

- ZYX Euler angle

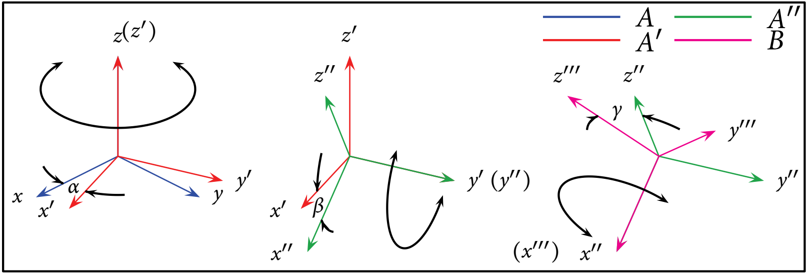

Figure 2.9

$$R_{aa'} = R_z(\alpha) \qquad R_{a'a''} = R_y(\beta) \qquad R_{a''b} = R_x(\gamma)$$

$$R_{ab} = R_z(\alpha)R_y(\beta)R_x(\gamma)$$

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2.2 Rotational Motion in $\mathbb{R}^3$

- *ZYX* Euler angle (continued)

$$R_{ab}(\alpha,\beta,\gamma) = \begin{bmatrix} c_\alpha c_\beta & -s_\alpha c_\gamma + c_\alpha s_\beta s_\gamma & s_\alpha s_\gamma + c_\alpha s_\beta c_\gamma \\ s_\alpha c_\beta & c_\alpha c_\gamma + s_\alpha s_\beta s_\gamma & -c_\alpha s_\gamma + s_\alpha s_\beta c_\gamma \\ -s_\beta & c_\beta s_\gamma & c_\beta c_\gamma \end{bmatrix}$$

Note: When $\beta = \frac{\pi}{2}$, $\cos\beta = 0$, $\alpha + \gamma = \text{const} \Rightarrow$ singularity!

$$\beta = \text{atan2}(-r_{31}, \sqrt{r_{32}^2 + r_{33}^2})$$
$$\alpha = \text{atan2}(r_{21}/c_\beta, r_{11}/c_\beta)$$
$$\gamma = \text{atan2}(r_{32}/c_\beta, r_{33}/c_\beta)$$

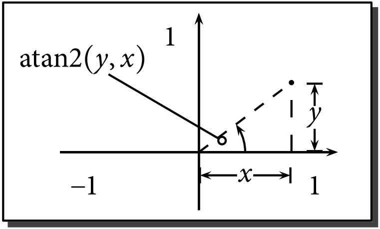
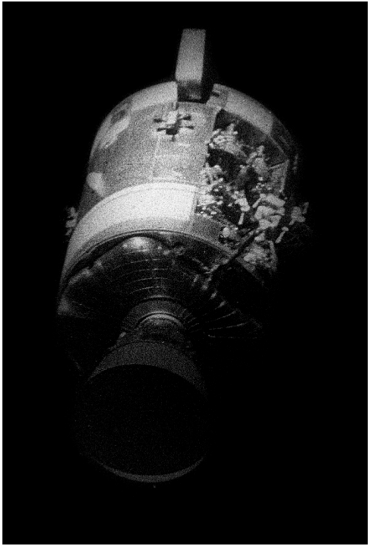


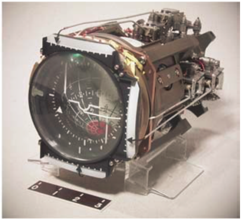
Figure 2.10

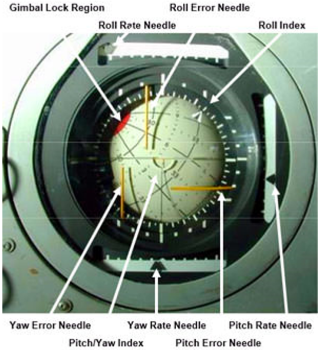
Apollo 10



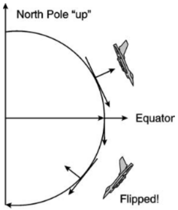

UPLINK ACTY
TEMP
NO ATT
GIMBAL LOCK
STBY
PROG
KEY REL
RESTART
OPR ERR
TRACKER
COMP ACTY
PROG
VERB
NOUN
7
8
9
CLR
ENTR
4
5
RSET
NOUN
1
2
3
KEY REL
©2000 Bruce M. Yaroro and the Smithsonian Institution


ABORT
LV ENGINES
GDC
TRANS CONTR PWR
OFF
LV TANK PRESS
©2000 Bruce M. Yarbro & The Smithsonian Institution




Gimbal Lock Region
Roll Error Needle
Roll Rate Needle
Roll Index
Yaw Error Needle
Yaw Rate Needle
Pitch Rate Needle
Pitch/Yaw Index
Pitch Error Needle

F-16 Fly-By-Wire Fighter Jet



Today

Chapter 2 Rigid Body Motion

2 Rotational motion in $\mathbb{R}^3$

- *Quaternions*

3 Rigid Motion in $\mathbb{R}^3$

- SE(3)
  - As a Configuration Space
  - *Homogeneous Representation*
  - *SE(3) is a Group*
  - *SE(3) is a Rigid Body Transformation*
- *Exponential coordinates of SE(3)*
  - *Twists*
  - *se(3)*
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- *Quaternions*

3 Rigid Motion in $\mathbb{R}^3$

- SE(3)
- *Exponential coordinates of SE(3)*

Ouaternions to the Rescue

Hamilton’s Walk

2.2 Rotational Motion in $\mathbb{R}^3$

§ Quaternions:

$Q = q_0 + q_1 i + q_2 j + q_3 k$
where $i^2 = j^2 = k^2 = -1, i \cdot j = k, j \cdot k = i, k \cdot i = j$

2.2 Rotational Motion in $\mathbb{R}^3$

§ Quaternions:

$Q = q_0 + q_1 i + q_2 j + q_3 k$
where $i^2 = j^2 = k^2 = -1, i \cdot j = k, j \cdot k = i, k \cdot i = j$

Property 1: Define $Q^* = (q_0, q)^* = (q_0, -q), q_0 \in \mathbb{R}, q \in \mathbb{R}^3$
$\|Q\|^2 = QQ^* = {q_0}^2 + {q_1}^2 + {q_2}^2 + {q_3}^2$

2.2 Rotational Motion in $\mathbb{R}^3$

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Property 2: $Q = (q_0, q), P = (p_0, p)$
$QP = (q_0 p_0 - q \cdot p, q_0 p + p_0 q + q \times p)$

2.2 Rotational Motion in $\mathbb{R}^3$

§ Quaternions:

$Q = q_0 + q_1 i + q_2 j + q_3 k$
where $i^2 = j^2 = k^2 = -1, i \cdot j = k, j \cdot k = i, k \cdot i = j$

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Property 2: $Q = (q_0, q), P = (p_0, p)$
$QP = (q_0 p_0 - q \cdot p, q_0 p + p_0 q + q \times p)$

Property 3: (a) The set of unit quaternions forms a group
(b) If $R = e^{\hat{\omega}\theta}$, then $Q = (\cos\frac{\theta}{2}, \omega \sin\frac{\theta}{2})$
(c) Q acts on $x \in \mathbb{R}^3$ by QXQ^*, where $X = (0, x)$

2.2 Rotational Motion in $\mathbb{R}^3$

□ Unit Quaternions:

Given $Q = (q_0, q), q_0 \in \mathbb{R}, q \in \mathbb{R}^3$, the vector part of QXQ^* is given by $R(Q)x$, recall that

$$q_0 = \cos\frac{\theta}{2}, q = \omega \sin\frac{\theta}{2}$$

and the Rodrigues' formula:

$$e^{\hat{\omega}\theta} = I + \hat{\omega}\sin\theta + \hat{\omega}^2(1 - \cos\theta)$$

then

$$\begin{aligned} R(Q) &= I + 2q_0\hat{q} + 2\hat{q}^2 \\ &= \begin{bmatrix} 1 - 2(q_2^2 + q_3^2) & -2q_0q_3 + 2q_1q_2 & 2q_0q_2 + 2q_1q_3 \\ 2q_0q_3 + 2q_1q_2 & 1 - 2(q_1^2 + q_3^2) & -2q_0q_1 + 2q_2q_3 \\ -2q_0q_2 + 2q_1q_3 & 2q_0q_1 + 2q_2q_3 & 1 - 2(q_1^2 + q_2^2) \end{bmatrix} \end{aligned}$$

where $\|Q\| \triangleq q_0^2 + q_1^2 + q_2^2 + q_3^2 = 1$

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□ Quaternions (continued):

Conversion from Roll-Pitch-Yaw angle to unit quaternions:

$$Q = (\cos\frac{\varphi}{2}, x\sin\frac{\varphi}{2})(\cos\frac{\theta}{2}, y\sin\frac{\theta}{2})(\cos\frac{\psi}{2}, z\sin\frac{\psi}{2}) \Rightarrow$$

$$q_0 = \cos\frac{\varphi}{2}\cos\frac{\theta}{2}\cos\frac{\psi}{2} - \sin\frac{\varphi}{2}\sin\frac{\theta}{2}\sin\frac{\psi}{2}$$

$$q = \begin{bmatrix} \cos\frac{\varphi}{2}\sin\frac{\theta}{2}\sin\frac{\psi}{2} + \sin\frac{\varphi}{2}\cos\frac{\theta}{2}\cos\frac{\psi}{2} \\ \cos\frac{\varphi}{2}\sin\frac{\theta}{2}\cos\frac{\psi}{2} - \sin\frac{\varphi}{2}\cos\frac{\theta}{2}\sin\frac{\psi}{2} \\ \cos\frac{\varphi}{2}\cos\frac{\theta}{2}\sin\frac{\psi}{2} + \sin\frac{\varphi}{2}\sin\frac{\theta}{2}\cos\frac{\psi}{2} \end{bmatrix}$$

Conversion from unit quaternions to roll-pitch-yaw angles (?)

† End of Section †

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- *Exponential coordinates of SE(3)*
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  - *se(3)*
  - *The Exponential Map*

2.3 Rigid motion in $\mathbb{R}^3$

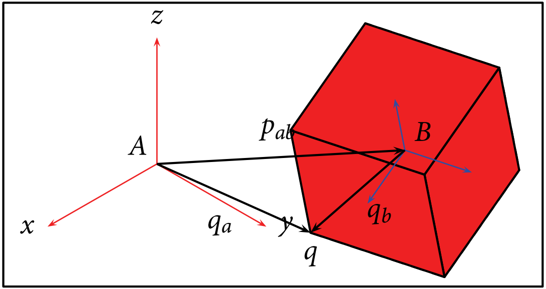

Figure 2.11

$p_{ab} \in \mathbb{R}^3$: Coordinates of the origin of B

$R_{ab} \in SO(3)$: Orientation of B relative to A

$SE(3) : \{(p, R) | p \in \mathbb{R}^3, R \in SO(3)\}$: Configuration Space

2.3 Rigid motion in $\mathbb{R}^3$

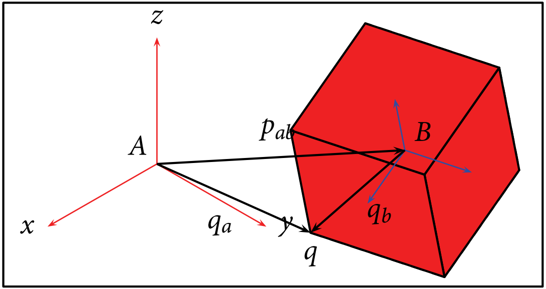

Figure 2.11

$p_{ab} \in \mathbb{R}^3$: Coordinates of the origin of B

$R_{ab} \in SO(3)$: Orientation of B relative to A

$SE(3) : \{(p, R) | p \in \mathbb{R}^3, R \in SO(3)\}$: Configuration Space

Or...as a transformation:

$$g_{ab} = (p_{ab}, R_{ab}) : \mathbb{R}^3 \mapsto \mathbb{R}^3$$

$$q_b \mapsto q_a = p_{ab} + R_{ab} \cdot q_b$$

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2.3 Rigid motion in $\mathbb{R}^3$

□ Homogeneous Representation:

Points:

$$q = \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} \in \mathbb{R}^3 \quad \Rightarrow \quad \bar{q} = \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ 1 \end{bmatrix} \in \mathbb{R}^4$$

□ Homogeneous Representation:

Points:

$$q = \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} \in \mathbb{R}^3 \quad \Rightarrow \quad \bar{q} = \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ 1 \end{bmatrix} \in \mathbb{R}^4$$

Vectors:

$$v = p - q = \begin{bmatrix} p_1 - q_1 \\ p_2 - q_2 \\ p_3 - q_3 \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \quad \Rightarrow \quad \bar{v} = \bar{p} - \bar{q} = \begin{bmatrix} p_1 \\ p_2 \\ p_3 \\ 1 \end{bmatrix} - \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ 1 \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ 0 \end{bmatrix}$$

2.3 Rigid motion in $\mathbb{R}^3$

☐ Homogeneous Representation:

Points:

$$q = \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} \in \mathbb{R}^3 \quad \Rightarrow \quad \bar{q} = \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ 1 \end{bmatrix} \in \mathbb{R}^4$$

Vectors:

$$v = p - q = \begin{bmatrix} p_1 - q_1 \\ p_2 - q_2 \\ p_3 - q_3 \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \quad \Rightarrow \quad \bar{v} = \bar{p} - \bar{q} = \begin{bmatrix} p_1 \\ p_2 \\ p_3 \\ 1 \end{bmatrix} - \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ 1 \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ 0 \end{bmatrix}$$

1. Point-Point = Vector
2. Vector+Point = Point
3. Vector+Vector = Vector
4. Point+Point: Meaningless

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2.3 Rigid motion in $\mathbb{R}^3$

$$q_a = p_{ab} + R_{ab} \cdot q_b$$

$$\begin{bmatrix} q_a \\ 1 \end{bmatrix} = \underbrace{\begin{bmatrix} R_{ab} & p_{ab} \\ 0 & 1 \end{bmatrix}}_{\bar{g}_{ab}} \begin{bmatrix} q_b \\ 1 \end{bmatrix}$$

$$\bar{q}_a = \bar{g}_{ab} \cdot \bar{q}_b$$

$$g_{ab} = (p_{ab}, R_{ab})$$

↓

$$\bar{g}_{ab} = \begin{bmatrix} R_{ab} & p_{ab} \\ 0 & 1 \end{bmatrix}$$

2.3 Rigid motion in $\mathbb{R}^3$

$$q_a = p_{ab} + R_{ab} \cdot q_b$$

$$\begin{bmatrix} q_a \\ 1 \end{bmatrix} = \underbrace{\begin{bmatrix} R_{ab} & p_{ab} \\ 0 & 1 \end{bmatrix}}_{\bar{g}_{ab}} \begin{bmatrix} q_{ab} \\ 1 \end{bmatrix}$$

$$\bar{q}_a = \bar{g}_{ab} \cdot \bar{q}_b$$

$$g_{ab} = (p_{ab}, R_{ab})$$

$$\bar{g}_{ab} = \begin{bmatrix} R_{ab} & p_{ab} \\ 0 & 1 \end{bmatrix}$$

□ Composition Rule:

$$\bar{q}_b = \bar{g}_{bc} \cdot \bar{q}_c$$
$$\bar{q}_a = \bar{g}_{ab} \cdot \bar{q}_b$$

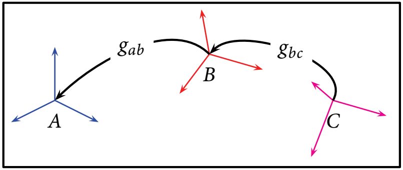

Figure 2.12

2.3 Rigid motion in $\mathbb{R}^3$

$$q_a = p_{ab} + R_{ab} \cdot q_b$$

$$\begin{bmatrix} q_a \\ 1 \end{bmatrix} = \underbrace{\begin{bmatrix} R_{ab} & p_{ab} \\ 0 & 1 \end{bmatrix}}_{\bar{g}_{ab}} \begin{bmatrix} q_{ab} \\ 1 \end{bmatrix}$$

$$\bar{q}_a = \bar{g}_{ab} \cdot \bar{q}_b$$

$$g_{ab} = (p_{ab}, R_{ab})$$

$$\bar{g}_{ab} = \begin{bmatrix} R_{ab} & p_{ab} \\ 0 & 1 \end{bmatrix}$$

□ Composition Rule:

$$\bar{q}_b = \bar{g}_{bc} \cdot \bar{q}_c$$

$$\bar{q}_a = \bar{g}_{ab} \cdot \bar{q}_b = \underbrace{\bar{g}_{ab} \cdot \bar{g}_{bc}}_{\bar{g}_{ac}} \cdot \bar{q}_c$$

$$\bar{g}_{ac} = \bar{g}_{ab} \cdot \bar{g}_{bc} = \begin{bmatrix} R_{ab}R_{bc} & R_{ab}p_{bc} + p_{ab} \\ 0 & 1 \end{bmatrix}$$

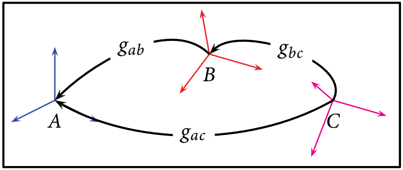

Figure 2.12

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2.3 Rigid motion in $\mathbb{R}^3$

□ Special Euclidean Group:

$$SE(3) = \left\{ \begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix} \in \mathbb{R}^{4\times 4} \middle| p \in \mathbb{R}^3, R \in SO(3) \right\}$$

2.3 Rigid motion in $\mathbb{R}^3$

Special Euclidean Group:

$$SE(3) = \left\{ \begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix} \in \mathbb{R}^{4\times4} \middle| p \in \mathbb{R}^3, R \in SO(3) \right\}$$

Property 4: $SE(3)$ forms a group.

2.3 Rigid motion in $\mathbb{R}^3$

Special Euclidean Group:

$$SE(3) = \left\{ \begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix} \in \mathbb{R}^{4\times 4} \middle| p \in \mathbb{R}^3, R \in SO(3) \right\}$$

Property 4: $SE(3)$ forms a group.

Proof :

1. $g_1 \cdot g_2 \in SE(3)$
2. $e = I_4$
3. $(\bar{g})^{-1} = \begin{bmatrix} R^T & -R^T p \\ 0 & 1 \end{bmatrix}$
4. Associativity: Follows from property of matrix multiplication

□

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§ Induced transformation on vectors:

$$\bar{v} = s - r = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ 0 \end{bmatrix}, \bar{g}_* \bar{v} = \bar{g}s - \bar{g}r = \begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ 0 \end{bmatrix} = \begin{bmatrix} Rv \\ 0 \end{bmatrix}$$

The bar will be dropped to simplify notations

Property 5: An element of $SE(3)$ is a rigid transformation.

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2.3 Rigid motion in $\mathbb{R}^3$

Exponential coordinates of $SE(3)$:

For rotational motion:

$$\dot{p}(t) = \omega \times (p(t) - q)$$

$$\begin{bmatrix} \dot{p} \\ 0 \end{bmatrix} = \begin{bmatrix} \hat{\omega} & -\omega \times q \\ 0 & 0 \end{bmatrix} \begin{bmatrix} p \\ 1 \end{bmatrix}$$

or $\dot{\bar{p}} = \hat{\xi} \cdot \bar{p} \Rightarrow \bar{p}(t) = e^{\hat{\xi} t} \bar{p}(0)$

where $e^{\hat{\xi} t} = I + \hat{\xi} t + \frac{(\hat{\xi} t)^2}{2!} + \cdots$

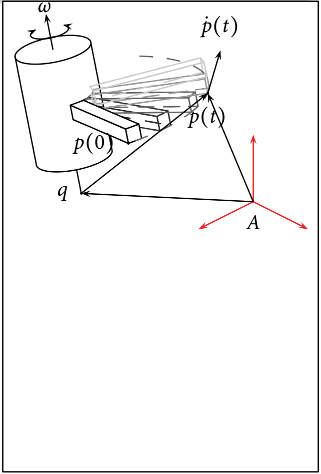

Figure 2.13

2.3 Rigid motion in $\mathbb{R}^3$

Exponential coordinates of $SE(3)$:

For rotational motion:

$$\dot{p}(t) = \omega \times (p(t) - q)$$

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$$\text{or } \dot{\bar{p}} = \hat{\xi} \cdot \bar{p} \Rightarrow \bar{p}(t) = e^{\hat{\xi}t} \bar{p}(0)$$

$$\text{where } e^{\hat{\xi}t} = I + \hat{\xi}t + \frac{(\hat{\xi}t)^2}{2!} + \cdots$$

For translational motion:

$$\dot{p}(t) = v$$

$$\begin{bmatrix} \dot{p}(t) \\ 0 \end{bmatrix} = \begin{bmatrix} 0 & v \\ 0 & 0 \end{bmatrix} \begin{bmatrix} p \\ 1 \end{bmatrix}$$

$$\dot{\bar{p}}(t) = \hat{\xi} \cdot \bar{p}(t) \Rightarrow \bar{p}(t) = e^{\hat{\xi}t} \bar{p}(0)$$

$$\hat{\xi} = \begin{bmatrix} 0 & v \\ 0 & 0 \end{bmatrix}$$

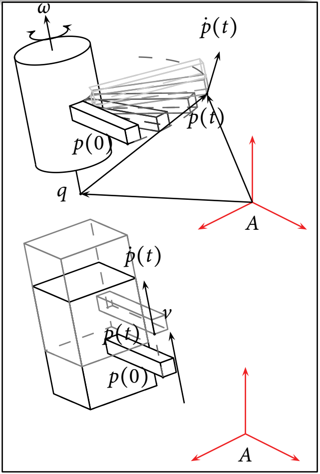

Figure 2.13

Today

Chapter 2 Rigid Body Motion

2 Rotational motion in $\mathbb{R}^3$

- *Quaternions*

3 Rigid Motion in $\mathbb{R}^3$

- SE(3)
  - As a Configuration Space
  - *Homogeneous Representation*
  - *SE(3) is a Group*
  - *SE(3) is a Rigid Body Transformation*
- *Exponential coordinates of SE(3)*
  - *Twists*
  - *se(3)*
  - *The Exponential Map*

2.3 Rigid motion in $\mathbb{R}^3$

Definition:

$$se(3) = \left\{ \begin{bmatrix} \hat{\omega} & v \\ 0 & 0 \end{bmatrix} \in \mathbb{R}^{4\times 4} \middle| v, \omega \in \mathbb{R}^3 \right\}$$

is called the twist space. There exists a 1-1 correspondence between $se(3)$ and $\mathbb{R}^6$, defined by $\wedge : \mathbb{R}^6 \mapsto se(3)$

$$\xi := \begin{bmatrix} v \\ \omega \end{bmatrix} \mapsto \hat{\xi} = \begin{bmatrix} \hat{\omega} & v \\ 0 & 0 \end{bmatrix}$$

Today

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Proof :

Let $\hat{\xi} = \begin{bmatrix} \hat{\omega} & v \\ 0 & 0 \end{bmatrix}$

- If $\omega = 0$, then $\hat{\xi}^2 = \hat{\xi}^3 = \cdots = 0$, $e^{\hat{\xi}\theta} = \begin{bmatrix} I & v\theta \\ 0 & 1 \end{bmatrix} \in SE(3)$

(continues next slide)

2.3 Rigid motion in $\mathbb{R}^3$

- If ω is not 0, assume $\|\omega\| = 1$.

Define:

$$g_0 = \begin{bmatrix} I & \omega \times v \\ 0 & 1 \end{bmatrix}, \hat{\xi}' = g_0^{-1} \cdot \hat{\xi} \cdot g_0 = \begin{bmatrix} \hat{\omega} & h\omega \\ 0 & 0 \end{bmatrix}$$

where $h = \omega^T \cdot v$.

$$e^{\hat{\xi}\theta} = e^{g_0 \cdot \hat{\xi}' \cdot g_0^{-1}} = g_0 \cdot e^{\hat{\xi}'\theta} \cdot g_0^{-1}$$

and as

$$\hat{\xi}'^2 = \begin{bmatrix} \hat{\omega}^2 & 0 \\ 0 & 0 \end{bmatrix}, \hat{\xi}'^3 = \begin{bmatrix} \hat{\omega}^3 & 0 \\ 0 & 0 \end{bmatrix}$$

we have

$$e^{\hat{\xi}'\theta} = \begin{bmatrix} e^{\hat{\omega}\theta} & h\omega\theta \\ 0 & 1 \end{bmatrix} \Rightarrow e^{\hat{\xi}\theta} = \begin{bmatrix} e^{\hat{\omega}\theta} & (I - e^{\hat{\omega}\theta})\hat{\omega}v + \omega\omega^T v\theta \\ 0 & 1 \end{bmatrix}$$

□

2.3 Rigid motion in $\mathbb{R}^3$

$$p(\theta) = e^{\hat{\xi}\theta} \cdot p(0) \Rightarrow g_{ab}(\theta) = e^{\hat{\xi}\theta}$$

If there is offset,

$$g_{ab}(\theta) = e^{\hat{\xi}\theta} g_{ab}(0)(\text{ Why?})$$

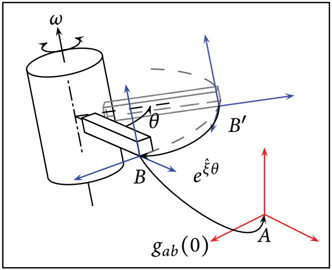

Figure 2.14

2.3 Rigid motion in $\mathbb{R}^3$

Property 7: $exp : se(3) \mapsto SE(3)$ is onto.

2.3 Rigid motion in $\mathbb{R}^3$

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Proof :

Let $g = (p, R), R \in SO(3), p \in \mathbb{R}^3$

Case 1: $(R = I)$ Let

$$\hat{\xi} = \begin{bmatrix} 0 & \frac{p}{\|p\|} \\ 0 & 0 \end{bmatrix}, \theta = \|p\| \Rightarrow e^{\hat{\xi}\theta} = g = \begin{bmatrix} I & p \\ 0 & 1 \end{bmatrix}$$

2.3 Rigid motion in $\mathbb{R}^3$

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Case 2: $(R \neq I)$

$$e^{\hat{\xi}\theta} = \begin{bmatrix} e^{\hat{\omega}\theta} & (I - e^{\hat{\omega}\theta})(\omega \times v) + \omega\omega^T v\theta \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{cases} e^{\hat{\omega}\theta} = R \\ (I - e^{\hat{\omega}\theta})(\omega \times v) + \omega\omega^T v\theta = p \end{cases}$$

Solve for $\omega\theta$ from previous section. Let $A = (I - e^{\hat{\omega}\theta})\hat{\omega} + ww^T\theta$, $Av = p$. Claim:

$$A = (I - e^{\hat{\omega}\theta})\hat{\omega} + ww^T\theta := A_1 + A_2$$

$$\ker A_1 \cap \ker A_2 = \phi \Rightarrow v = A^{-1}p$$

□

$\xi\theta \in \mathbb{R}^6$: Exponential coordinates of $g \in SE(3)$