Last Time

Chapter 2 Rigid Body Motion

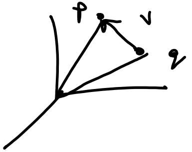
1 Rigid Body Transformations

- Length Preserving: $||g(p) - g(q)|| = ||p - q||$
- Orientation Preserving: $g_*(v \times w) = g_*(v) \times g_*(w)$

2 Rotational motion in $\mathbb{R}^3$

- Rotation Matrix
  - Represents *configuration*
  - Represents *(rotational) transformation*
- Rotation Matrices with matrix multiplication form a *Group*
- Rotational Transformation is a *Rigid Body Transformation*

Recap



- Rigid body Transformations

  Point $p = \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} \in \mathbb{R}^3$

  Vector $v = p - q = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \in \mathbb{R}^3$

  Trajectory $p(t) : \mathbb{R} \longrightarrow \mathbb{R}^3$

  $t \longmapsto \begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix}$

Properties of Rigid body Transformations :

(i) $\|g(p) - g(q)\| = \|p - q\|$ — Preserves length

(ii) $g_*(v \times w) = g_*(v) \times g_*(w)$

- Rotational Motion :

  Rotation Matrix $R_{ab} = \begin{bmatrix} x_{ab} & y_{ab} & z_{ab} \end{bmatrix}$

  - Represents config. of Body Frame "B" wrt spatial frame "A"

- Represents (rotational) transformation to convert vectors in "B" to "A"

$$q_a = R_{ab} \, q_b$$

$$SO(3) = \{ R \in \mathbb{R}^{3\times3} \mid \underbrace{R^T R = I}_{\text{orthogonal}}, \ \underbrace{\det(R) = +1}_{\text{special}} \}$$

$$\Downarrow$$

$$\det(R) = \pm 1$$

- $(SO(3), \cdot)$ is a Group :
  - (i) Closure : $R_1 R_2 \in SO(3) \implies R_1 R_2 \in SO(3)$
  - (ii) Identity element : $I \in SO(3)$ s.t. $R \cdot I = R \in SO(3)$
  - (iii) Inverse element : $\forall R \in SO(3)$, $R^{-1} = R^T$ s.t. $RR^T = I$
  - (iv) Associativity

- Rotational Motion is a Rigid body Transformation

(i) Preserves length : $\| R_{ab} \cdot (P_b - P_a) \| = \| P_b - P_a \|$

(ii) Preserves orientation : $R\,(v \times w) = (Rv) \times (Rw)$

used property $R \hat{v} R^T = \widehat{Rv}$

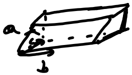
$so(3) = \{ A \in \mathbb{R}^{3\times 3} \mid A^T = -A \}$

hat map : $\wedge : \mathbb{R}^3 \longrightarrow so(3)$

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \stackrel{\wedge}{\longmapsto} \begin{bmatrix} 0 & -v_3 & v_2 \\ v_3 & 0 & -v_1 \\ -v_2 & v_1 & 0 \end{bmatrix}$$

$v \times w = \hat{v}\, w$

volume



(1) scalar triple product : $a \cdot (b \times c) = c \cdot (a \times b) = b \cdot (c \times a)$

(2) Vector triple product : $a \times (b \times c) = (a \cdot c)\, b - (a \cdot b)\, c$

(3) Jacobi Identity : $(a \times b) \times c + (c \times a) \times b + (b \times c) \times a = 0$

Today

Chapter 2 Rigid Body Motion

2 Rotational motion in $\mathbb{R}^3$

- The *Exponential Map*
- Rodrigues *Formula*
- *Euler Angles*
- *Quaternions*

2.2 Rotational Motion in $\mathbb{R}^3$

□ Parametrization of $SO(3)$ (the exponential coordinate):

◇ **Review:** $S^1 = \{z \in \mathbb{C} \,|\, |z| = 1\}$

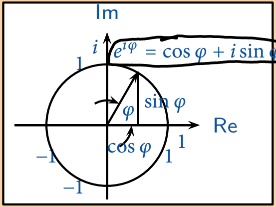


Figure 2.4

Euler's Formula

"One of the most remarkable, almost astounding, formulas in all of mathematics."

R. Feynman

$i = \sqrt{-1}$

2.2 Rotational Motion in $\mathbb{R}^3$

□ Parametrization of $SO(3)$ (the exponential coordinate):

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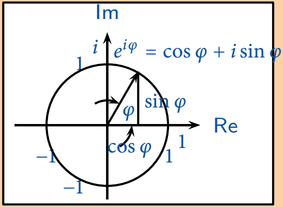

Figure 2.4

Euler's Formula

"One of the most remarkable, almost astounding, formulas in all of mathematics."

R. Feynman

◇ **Review:**

$x(t), x_0, a \in \mathbb{R}$

$x_0, x(t) \in \mathbb{R}^n$, $A \in \mathbb{R}^{n\times n}$

(1) ODE

$\dot{x} = \frac{dx}{dt}$

$$\begin{cases} \dot{x}(t) = ax(t) \\ x(0) = x_0 \end{cases} \Rightarrow \boxed{x(t) = e^{at}x_0}$$

$\dot{x}(t) = A\,x(t)$, $x(0) = x_0$

$x(t) = e^{At}x_0$

$e^{At} = I + At + \frac{A^2t^2}{2!} + \cdots$

Matrix exponential

(Continues next slide)

2.2 Rotational Motion in $\mathbb{R}^3$

(2) constraints on R

$$R \in SO(3), R = \begin{array}{c} \begin{array}{ccc} r_1 & r_2 & r_3 \end{array} \\ \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} \end{array}$$

$$r_i \cdot r_j = \begin{cases} 0, & i \neq j \\ 1, & i = j \end{cases} \quad \leftarrow \text{6 constraints}$$

$$\Rightarrow \boxed{\text{3 independent parameters!}}$$

2.2 Rotational Motion in $\mathbb{R}^3$

$$R \in SO(3), R = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}$$

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$\Rightarrow$ 3 independent parameters!

Rotational vel. of Point.

(3) Consider motion of a point q on a rotating link

$$\begin{cases} \boxed{\dot{q}(t) = \omega \times q(t)} = \hat{\omega} q(t) \\ q(0)\text{: Initial coordinates} \end{cases}$$

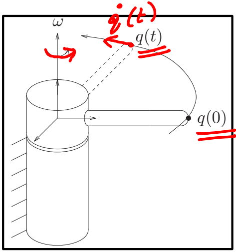

Figure 2.5

2.2 Rotational Motion in $\mathbb{R}^3$

$$R \in SO(3), R = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}$$

$$r_i \cdot r_j = \begin{cases} 0, & i \neq j \\ 1, & i = j \end{cases} \quad \leftarrow \text{6 constraints}$$

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Consider motion of a point q on a rotating link

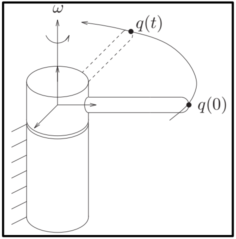

Figure 2.5

$$\begin{cases} \dot{q}(t) = \omega \times q(t) = \hat{\omega} q(t) \\ q(0)\text{: Initial coordinates} \end{cases}$$

$$\Rightarrow \boxed{q(t) = e^{\hat{\omega} t} q_0} \text{ where } e^{\hat{\omega} t} = I + \hat{\omega} t + \frac{(\hat{\omega} t)^2}{2!} + \frac{(\hat{\omega} t)^3}{3!} + \cdots$$

2.2 Rotational Motion in $\mathbb{R}^3$

$$R \in SO(3), R = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}$$

$$r_i \cdot r_j = \begin{cases} 0, & i \neq j \\ 1, & i = j \end{cases} \quad \leftarrow \text{6 constraints}$$

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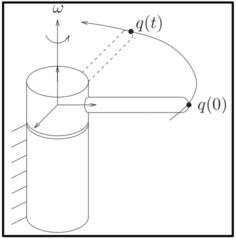

Figure 2.5

Consider motion of a point q on a rotating link

$$\begin{cases} \dot{q}(t) = \underline{\omega} \times q(t) = \hat{\omega} q(t) \\ q(0)\text{: Initial coordinates} \end{cases}$$

$$\Rightarrow q(t) = e^{\hat{\omega} t} q_0 \text{ where } \boxed{\boxed{e^{\hat{\omega} t}} = I + \hat{\omega} t + \frac{(\hat{\omega} t)^2}{2!} + \frac{(\hat{\omega} t)^3}{3!} + \cdots}$$

By the definition of rigid transformation, $R(\underline{\omega}, \underline{\theta}) = \boxed{e^{\hat{\omega}\theta}}$. Let $so(3) = \{\hat{\omega} | \omega \in \mathbb{R}^3\}$ or $so(n) = \{S \in \mathbb{R}^{n \times n} | S^T = -S\}$ where $\wedge : \mathbb{R}^3 \mapsto so(3) : \omega \mapsto \hat{\omega}$, we have:

Exponential map

$\exp : so(3) \longrightarrow SO(3)$

$\hat{\omega}\theta \longrightarrow e^{\hat{\omega}\theta}$

Property 3: $\exp : so(3) \mapsto SO(3), \hat{\omega}\theta \mapsto e^{\hat{\omega}\theta}$

Today

Chapter 2 Rigid Body Motion

2 Rotational motion in $\mathbb{R}^3$

- The *Exponential Map*
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2.2 Rotational Motion in $\mathbb{R}^3$

Rodrigues' formula ($\|\omega\| = 1$):

$$e^{\hat{\omega}\theta} = I + \hat{\omega}\sin\theta + \hat{\omega}^2(1 - \cos\theta)$$

Proof: $LHS = e^{\hat{\omega}\theta} = I + \hat{\omega}\theta + \frac{\hat{\omega}^2\theta^2}{2!} + \frac{\hat{\omega}^3\theta^3}{3!} + \cdots$

$$\hat{\omega} = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix}$$ Skew-symmetric

$$\hat{\omega}^2 = \begin{bmatrix} -(\omega_3^2+\omega_2^2) & \omega_1\omega_2 & \omega_1\omega_3 \\ \omega_1\omega_2 & -(\omega_1^2+\omega_3^2) & \omega_2\omega_3 \\ \omega_1\omega_3 & \omega_2\omega_3 & -(\omega_2^2+\omega_1^2) \end{bmatrix}$$ symmetric

$$= \begin{bmatrix} \omega_1^2 & \omega_1\omega_2 & \omega_1\omega_3 \\ \omega_1\omega_2 & \omega_2^2 & \omega_2\omega_3 \\ \omega_1\omega_3 & \omega_2\omega_3 & \omega_3^2 \end{bmatrix} - (\omega_1^2 + \omega_2^2 + \omega_3^2) I$$

$$\boxed{\hat{\omega}^2 = \omega\omega^T - I}$$

$\square$

$$\hat{\omega}^3 = \hat{\omega}(\omega\omega^T - I) = \underbrace{\hat{\omega}\,\omega}_{=0}\omega^T - \hat{\omega} = -\hat{\omega}$$

$$\hat{\omega}^4 = \hat{\omega}\,\hat{\omega}^3 = \hat{\omega}(-\hat{\omega}) = -\hat{\omega}^2$$

$$\hat{\omega}^5 = \hat{\omega}\,\hat{\omega}^4 = -\hat{\omega}^3 = \hat{\omega}$$

$$e^{\hat{\omega}\theta} = I + \hat{\omega}\left[\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots\right] + \hat{\omega}^2\left[\frac{\theta^2}{2!} - \frac{\theta^4}{4!} + \frac{\theta^6}{6!} - \cdots\right]$$

$$\boxed{e^{\hat{\omega}\theta} = I + \hat{\omega}\sin\theta + \hat{\omega}^2(1 - \cos\theta)}$$ — Rodrigues Formula

2.2 Rotational Motion in $\mathbb{R}^3$

Rodrigues' formula ($\|\omega\| = 1$):

$$e^{\hat{\omega}\theta} = I + \hat{\omega}\sin\theta + \hat{\omega}^2(1-\cos\theta)$$

Proof :

Let $a \in \mathbb{R}^3$, write

$$a = \omega\theta, \omega = \frac{a}{\|a\|} (\text{or } \|\omega\| = 1), \text{and } \theta = \|a\|$$

$$e^{\hat{\omega}\theta} = I + \hat{\omega}\theta + \frac{(\hat{\omega}\theta)^2}{2!} + \frac{(\hat{\omega}\theta)^3}{3!} + \cdots$$

As

$$\hat{a}^2 = aa^T - \|a\|^2 I, \hat{a}^3 = -\|a\|^2\hat{a}$$

we have:

$$\begin{aligned} e^{\hat{\omega}\theta} &= I + (\theta - \frac{\theta^3}{3!} + \frac{\theta^3}{5!} - \cdots)\hat{\omega} + (\frac{\theta^2}{2!} - \frac{\theta^4}{4!} + \cdots)\hat{\omega}^2 \\ &= I + \hat{\omega}\sin\theta + \hat{\omega}^2(1-\cos\theta) \end{aligned}$$

$\square$

2.2 Rotational Motion in $\mathbb{R}^3$

Rodrigues' formula for $\|\omega\| \neq 1$:

$$e^{\hat{\omega}\theta} = I + \frac{\hat{\omega}}{\|\omega\|}\sin\|\omega\|\theta + \frac{\hat{\omega}^2}{\|\omega\|^2}(1 - \cos\|\omega\|\theta)$$

2.2 Rotational Motion in $\mathbb{R}^3$

Rodrigues' formula for $\|\omega\| \neq 1$:

$$e^{\hat{\omega}\theta} = I + \frac{\hat{\omega}}{\|\omega\|}\sin\|\omega\|\theta + \frac{\hat{\omega}^2}{\|\omega\|^2}(1 - \cos\|\omega\|\theta)$$

Proof for Property 3:

$\hat{\omega}$ skew symmetric

Let $\underline{R \triangleq e^{\hat{\omega}\theta}}$, then:

$$(e^{\hat{\omega}\theta})^{-1} = e^{-\hat{\omega}\theta} = e^{\hat{\omega}^T\theta} = (e^{\hat{\omega}\theta})^T$$

$$\Rightarrow \underline{R^{-1} = R^T} \Rightarrow \underline{R^T R = I \Rightarrow \det R = \pm 1}$$

det I

From $\boxed{\det\exp(0) = 1}$ and the continuity of det function w.r.t. θ, we have $\boxed{\det e^{\hat{\omega}\theta} = 1,}\ \forall\theta \in \mathbb{R}$ □

$$\mathbb{R}^3 \underset{\vee\ \text{(vee map)}}{\overset{\wedge}{\rightleftarrows}} so(3) \underset{\log}{\overset{\exp}{\rightleftarrows}} SO(3)$$

2.2 Rotational Motion in $\mathbb{R}^3$

Property 4: The exponential map is onto.

2.2 Rotational Motion in $\mathbb{R}^3$

$\exp: so(3) \longrightarrow SO(3)$

Property 4: The exponential map is onto.

Proof :

Given $R \in SO(3)$, to show $\exists \omega \in \mathbb{R}^3, \|\omega\| = 1$ and θ s.t. $\boxed{R = e^{\hat{\omega}\theta}}$

Let

Given
$$R = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}$$

and

$$v_\theta = 1 - \cos\theta, c_\theta = \cos\theta, s_\theta = \sin\theta$$

By Rodrigues' formula $e^{\hat{\omega}\theta} = I + \hat{\omega}\sin\theta + \hat{\omega}^2(1-\cos\theta)$

Recover θ, ω s.t. $e^{\hat{\omega}\theta} = R$

$$e^{\hat{\omega}\theta} = \begin{bmatrix} \omega_1^2 v_\theta + c_\theta & \omega_1\omega_2 v_\theta - \omega_3 s_\theta & \omega_1\omega_3 v_\theta + \omega_2 s_\theta \\ \omega_1\omega_2 v_\theta + \omega_3 s_\theta & \omega_2^2 v_\theta + c_\theta & \omega_2\omega_3 v_\theta - \omega_1 s_\theta \\ \omega_1\omega_3 v_\theta - \omega_2 s_\theta & \omega_2\omega_3 v_\theta + \omega_1 s_\theta & \omega_3^2 v_\theta + c_\theta \end{bmatrix}$$

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2.2 Rotational Motion in $\mathbb{R}^3$

Taking the trace of both sides,

$$\mathrm{tr}(R) = r_{11} + r_{22} + r_{33} = 1 + 2\cos\theta = \sum_{i=1}^{3} \lambda_i \quad \in [-1, 3]$$

where λ_i is the eigenvalue of $R, i = 1, 2, 3$

Case 1: $\mathrm{tr}(R) = 3$ or $R = I$, $\theta = 0 \Rightarrow \omega\theta = 0$

2.2 Rotational Motion in $\mathbb{R}^3$

Taking the trace of both sides,

$$\mathrm{tr}(R) = r_{11} + r_{22} + r_{33} = \underline{1 + 2\cos\theta} = \sum_{i=1}^{3} \lambda_i$$

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Case 1: $\mathrm{tr}(R) = 3$ or $R = I,\ \theta = 0 \Rightarrow \omega\theta = 0$

Case 2: $\underline{-1 < \mathrm{tr}(R) < 3},$

$$\boxed{\theta = \arccos \frac{\mathrm{tr}(R) - 1}{2}} \Rightarrow \omega = \frac{1}{2s_\theta}\begin{bmatrix} r_{32} - r_{23} \\ r_{13} - r_{31} \\ r_{21} - r_{12} \end{bmatrix}$$

2.2 Rotational Motion in $\mathbb{R}^3$

Taking the trace of both sides,

$$\mathrm{tr}(R) = r_{11} + r_{22} + r_{33} = 1 + 2\cos\theta = \sum_{i=1}^{3} \lambda_i$$

where λ_i is the eigenvalue of $R, i = 1, 2, 3$

Case 1: $\mathrm{tr}(R) = 3$ or $R = I,\ \theta = 0 \Rightarrow \omega\theta = 0$

Case 2: $-1 < \mathrm{tr}(R) < 3,$

$$\theta = \arccos\frac{\mathrm{tr}(R) - 1}{2} \Rightarrow \omega = \frac{1}{2s_\theta}\begin{bmatrix} r_{32} - r_{23} \\ r_{13} - r_{31} \\ r_{21} - r_{12} \end{bmatrix}$$

Case 3: $\mathrm{tr}(R) = -1 \Rightarrow \cos\theta = -1 \Rightarrow \theta = \pm\pi$

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2.2 Rotational Motion in $\mathbb{R}^3$

$SO(3) \xrightarrow{\log} so(3)$

$R \longmapsto \hat{\omega}\,\theta$

$\xleftarrow{\exp}$

Following are 3 possibilities:

$$R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \Rightarrow \omega = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix},$$

$$R = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \Rightarrow \omega = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix},$$

$$R = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow \omega = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Note that if $\omega\theta$ is a solution, then $\omega(\theta \pm n\pi), n = 0, \pm1, \pm2, \ldots$ is also a solution. □

2.2 Rotational Motion in $\mathbb{R}^3$

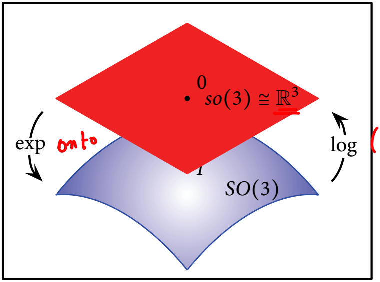
Definition: Exponential coordinate

$\omega\theta \in \mathbb{R}^3$, with $e^{\hat{\omega}\theta} = R$ is called the exponential coordinates of R

Exp :

$\mathbb{R}^{3\times3}$

$SO(3)$ $so(3)$



(1-1 for almost all point)

Figure 2.6

2.2 Rotational Motion in $\mathbb{R}^3$

Definition: Exponential coordinate

$\omega\theta \in \mathbb{R}^3$, with $e^{\hat{\omega}\theta} = R$ is called the exponential coordinates of R

$\text{Exp}:$

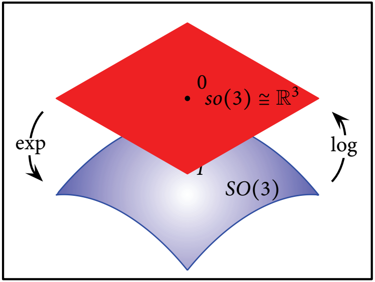


Figure 2.6

Property 5: exp is 1-1 when restricted to an open ball in $\mathbb{R}^3$ of radius π.

2.2 Rotational Motion in $\mathbb{R}^3$

Theorem 1 (Euler):
Any orientation is equivalent to a rotation about a fixed axis $\omega \in \mathbb{R}^3$ through an angle $\theta \in [-\pi, \pi]$.

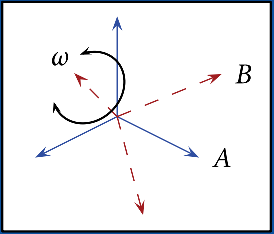


Figure 2.7

1707–1783

$\|\omega\| = 1, \ \theta$

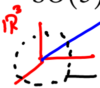
ω

$\exp\left(\widehat{\left(\frac{\omega}{\|\omega\|}\right)} \|\omega\|\right)$

θ

$SO(3)$ can be visualized as a solid ball of radius π.

ω



$\xrightarrow{\wedge} \hat{\omega} \xrightarrow{\exp} \exp(\hat{\omega}) \in SO(3)$

Radius π

Today

Chapter 2 Rigid Body Motion

2 Rotational motion in $\mathbb{R}^3$

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Other Parametrizations of $SO(3)$:

- XYZ fixed angles (or Roll-Pitch-Yaw angle)

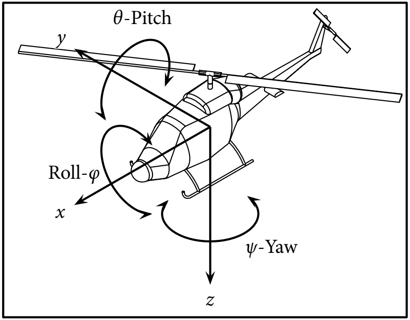

Figure 2.8

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2.2 Rotational Motion in $\mathbb{R}^3$

- XYZ fixed angles (or Roll-Pitch-Yaw angle) Continued

$$R_x(\varphi) := e^{\hat{x}\varphi} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\varphi & -\sin\varphi \\ 0 & \sin\varphi & \cos\varphi \end{bmatrix}$$

$$R_y(\theta) := e^{\hat{y}\theta} = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$$

$$\underline{R_z(\psi) := e^{\hat{z}\psi}} = \begin{bmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$R_{ab} = R_x(\varphi)R_y(\theta)R_z(\psi)$$
$$= \begin{bmatrix} c_\theta c_\psi & -c_\theta s_\psi & s_\theta \\ s_\varphi s_\theta c_\psi + c_\varphi s_\psi & -s_\varphi s_\theta s_\psi + c_\varphi c_\psi & -s_\varphi c_\theta \\ -c_\varphi s_\theta c_\psi + s_\varphi s_\psi & c_\varphi s_\theta s_\psi + s_\varphi c_\psi & c_\varphi c_\theta \end{bmatrix}$$

2.2 Rotational Motion in $\mathbb{R}^3$

(1) (2) (3)

- ZYX Euler angle

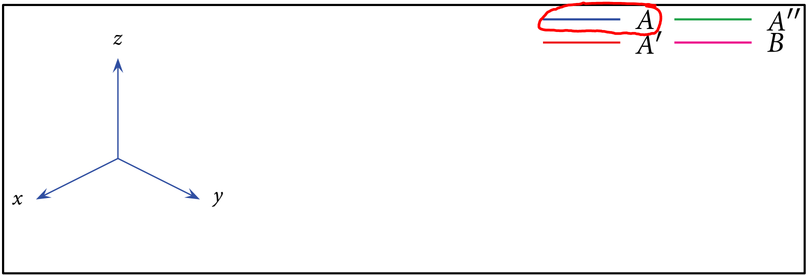


Figure 2.9

2.2 Rotational Motion in $\mathbb{R}^3$

- ZYX Euler angle

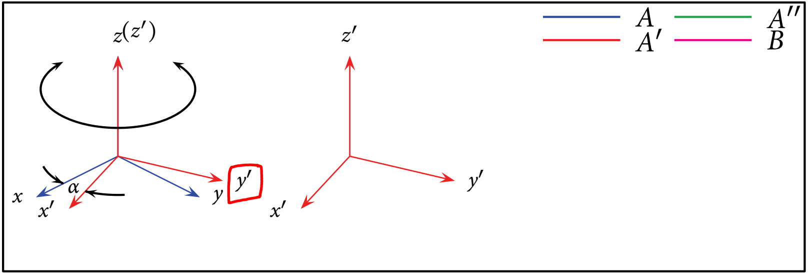


Figure 2.9

$$\boxed{R_{aa'} = R_z(\alpha)}$$

2.2 Rotational Motion in $\mathbb{R}^3$

- ZYX Euler angle

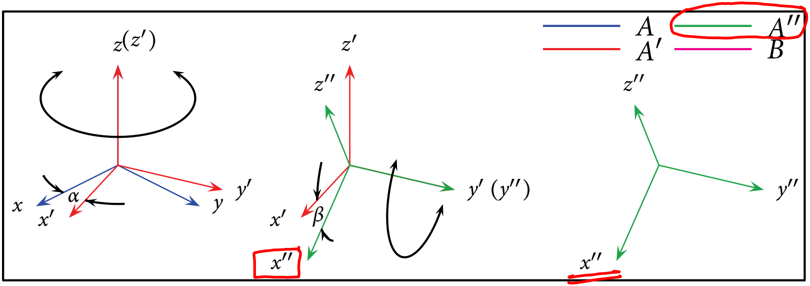


Figure 2.9

$$R_{aa'} = R_z(\alpha) \qquad R_{a'a''} = R_y(\beta)$$

2.2 Rotational Motion in $\mathbb{R}^3$

- ZYX Euler angle

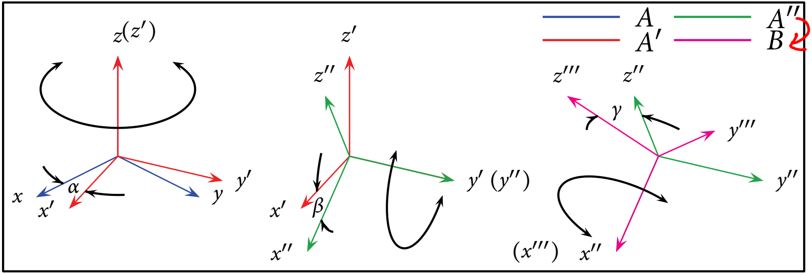


Figure 2.9

$$R_{aa'} = R_z(\alpha) \qquad R_{a'a''} = R_y(\beta) \qquad R_{a''b} = R_x(\gamma)$$

$$\boxed{R_{ab} = R_z(\alpha)R_y(\beta)R_x(\gamma)}$$

(continues next slide)

$$R_{ab} = R_{aa'} R_{a'a''} R_{a''b}$$

$$q_a = R_{ab} q_b$$

2.2 Rotational Motion in $\mathbb{R}^3$

- *ZYX* Euler angle (continued)

$$R_{ab}(\alpha,\beta,\gamma)=\begin{bmatrix} c_\alpha c_\beta & -s_\alpha c_\gamma + c_\alpha s_\beta s_\gamma & s_\alpha s_\gamma + c_\alpha s_\beta c_\gamma \\ s_\alpha c_\beta & c_\alpha c_\gamma + s_\alpha s_\beta s_\gamma & -c_\alpha s_\gamma + s_\alpha s_\beta c_\gamma \\ -s_\beta & c_\beta s_\gamma & c_\beta c_\gamma \end{bmatrix}$$

Note: When $\beta=\frac{\pi}{2}$, $\cos\beta=0$, $\alpha+\gamma=\text{const}\Rightarrow$ singularity!

$$\beta = \operatorname{atan2}(-r_{31}, \sqrt{r_{32}^2+r_{33}^2})$$
$$\alpha = \operatorname{atan2}(r_{21}/c_\beta, r_{11}/c_\beta)$$
$$\gamma = \operatorname{atan2}(r_{32}/c_\beta, r_{33}/c_\beta)$$

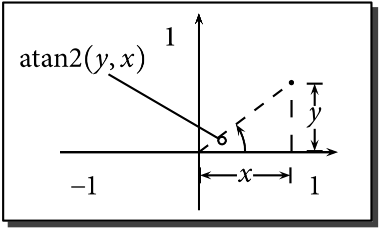
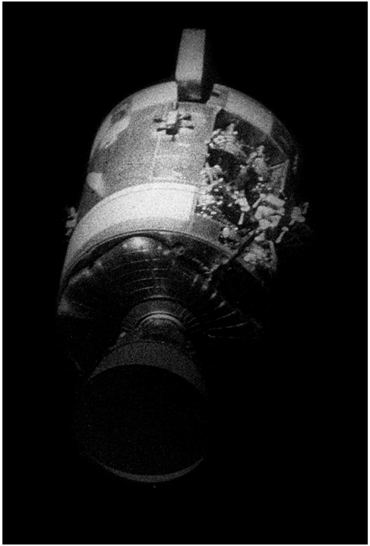


Figure 2.10

Apollo 10




UPLINK ACTY
TEMP
NO ATT
GIMBAL LOCK
STBY
PROG
KEY REL
RESTART
OPR ERR
TRACKER
COMP ACTY
PROG
VERB
NOUN
7
8
9
CLR
ENTR
–
4
5
NOUN
1
2
3
KEY REL
RSET
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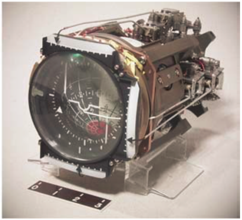
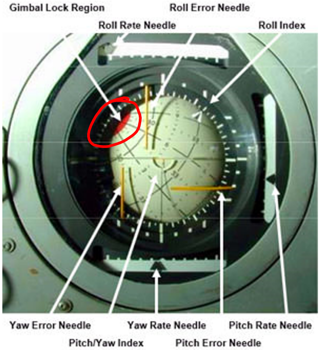
ABORT
©2000 Bruce M. Yarbro & The Smithsonian Institution




Gimbal Lock Region
Roll Error Needle
Roll Rate Needle
Roll Index
Yaw Error Needle
Yaw Rate Needle
Pitch Rate Needle
Pitch/Yaw Index
Pitch Error Needle