Last Time

Chapter 1 Robotics History

1. Robots and Robotics
2. Ancient History (3000 B.C.-1450 A.D.)
3. Early History (1451 A.D.-1960)
4. Modern History (1961-)
5. New Vistas

Today

Chapter 2 Rigid Body Motion

1. Rigid Body Transformations
2. Rotational motion in $\mathbb{R}^3$

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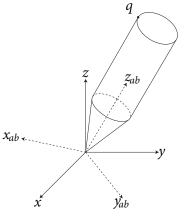
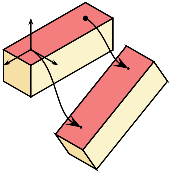
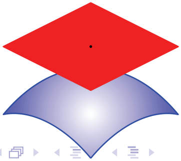
Chapter 2 Rigid Body Motion

1 Rigid Body Transformations

2 Rotational motion in $\mathbb{R}^3$







Today

Chapter 2 Rigid Body Motion

1 Rigid Body Transformations

- Length Preserving: $||g(p) - g(q)|| = ||p - q||$
- Orientation Preserving: $g_*(v \times w) = g_*(v) \times g_*(w)$

2 Rotational motion in $\mathbb{R}^3$

2.1 Rigid Body Transformations

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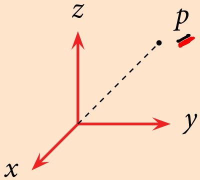
§ Notations:



$$p = \begin{bmatrix} p_x \\ p_y \\ p_z \end{bmatrix} \text{ or } p = \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix}$$

For $p \in \mathbb{R}^n, n = 2, 3$(2 for planar, 3 for spatial)

Point: $p = \begin{bmatrix} p_1 \\ p_2 \\ \vdots \\ p_n \end{bmatrix}, \; \|p\| = \sqrt{p_1^2 + \cdots + p_n^2}$

2.1 Rigid Body Transformations

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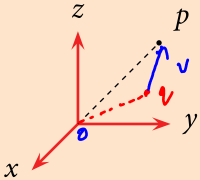

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Vector: $v = p - q = \begin{bmatrix} p_1 - q_1 \\ p_2 - q_2 \\ \vdots \\ p_n - q_n \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}, \; \|v\| = \sqrt{v_1^2 + \cdots + v_n^2}$

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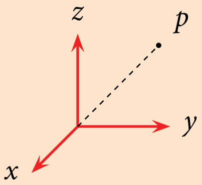

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Matrix: $A \in \mathbb{R}^{n \times m}, A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ \vdots & & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{bmatrix}$

2.1 Rigid Body Transformations

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□ Description of point-mass motion:

$$p(0) = \begin{bmatrix} x(0) \\ y(0) \\ z(0) \end{bmatrix}$$: initial position

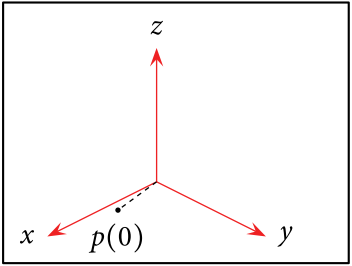

Figure 2.1

2.1 Rigid Body Transformations

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$$p(0) = \begin{bmatrix} x(0) \\ y(0) \\ z(0) \end{bmatrix}$$: initial position

$$p(t) = \begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix}, t \in (-\varepsilon, \varepsilon)$$

Trajectory: $t \in \mathbb{R} \longmapsto p(t) \in \mathbb{R}^3$

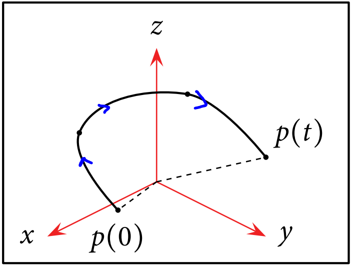

Figure 2.1

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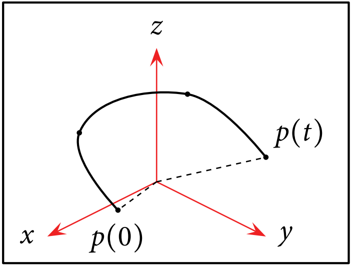

Figure 2.1

Definition: Trajectory

A **trajectory** is a curve $p : (-\varepsilon, \varepsilon) \mapsto \mathbb{R}^3, p(t) = \begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix}$

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☐ Rigid Body Motion:

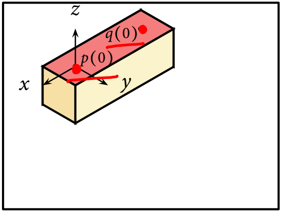

Figure 2.2

2.1 Rigid Body Transformations

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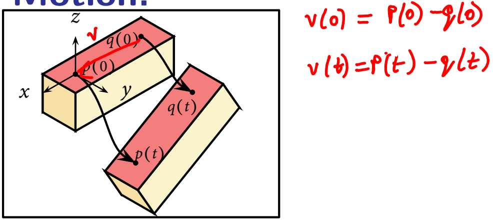

Figure 2.2

$$\|p(t) - q(t)\| = \|p(0) - q(0)\| = \text{constant}$$

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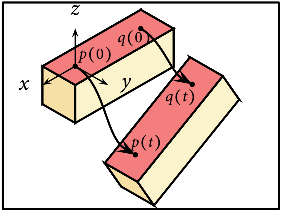

Figure 2.2

$$\|p(t) - q(t)\| = \|p(0) - q(0)\| = \text{constant}$$

Definition: Rigid body transformation

$$g : \mathbb{R}^3 \mapsto \mathbb{R}^3$$

s.t.

1. Length preserving: $\|g(p) - g(q)\| = \|p - q\|$
2. Orientation preserving: $g_*(v \times \omega) = g_*(v) \times g_*(\omega)$

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2 Rotational motion in $\mathbb{R}^3$

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Chapter 2 Rigid Body Motion

1 Rigid Body Transformations

2 Rotational motion in $\mathbb{R}^3$

- Rotation Matrix
  - Represents *configuration*
  - Represents *(rotational) transformation*
- Rotation Matrices with matrix multiplication form a *Group*
- Rotational Transformation is a *Rigid Body Transformation*

2.2 Rotational Motion in $\mathbb{R}^3$

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□ Rotational Motion:

1. Choose a reference frame A (spatial frame)

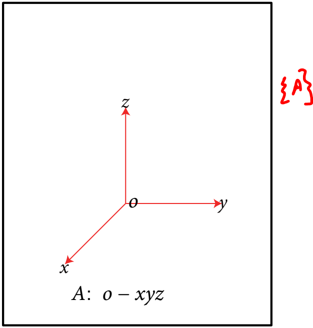

Figure 2.3

2.2 Rotational Motion in $\mathbb{R}^3$

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□ Rotational Motion:

1 Choose a reference frame A (spatial frame)
2 Attach a frame B to the body (body frame)

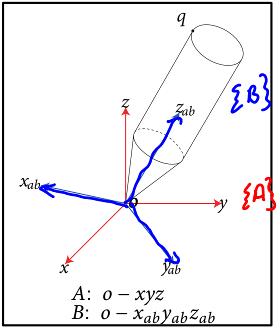

Figure 2.3

$$x_{ab} \in \mathbb{R}^3: \quad \text{coordinates of } x_b \text{ in frame } A$$

$$R_{ab} = [x_{ab}\ y_{ab}\ z_{ab}] \in \mathbb{R}^{3\times 3}: \quad \text{Rotation (or orientation) matrix of } B \text{ w.r.t. } A$$

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□ Property of a Rotation Matrix:

Let $R = [r_1 \; r_2 \; r_3]$ be a rotation matrix

2.2 Rotational Motion in $\mathbb{R}^3$

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□ Property of a Rotation Matrix:

Let $R = [r_1 \ r_2 \ r_3]$ be a rotation matrix

$$\Rightarrow r_i^T \cdot r_j = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$$

or $R^T \cdot R = \begin{bmatrix} r_1^T \\ r_2^T \\ r_3^T \end{bmatrix} [r_1 \ r_2 \ r_3] = I$ or $R \cdot R^T = I$

$r_1 \cdot r_2 = r_1^T r_2 = 0$

$r_1 \cdot r_1 = r_1^T r_1 = 1$

$O(3) = \{ R \in \mathbb{R}^{3\times3} \mid R^T R = R R^T = I \}$

$$R^T R = \begin{bmatrix} r_1^T r_1 & r_1^T r_2 & r_1^T r_3 \\ r_2^T r_1 & r_2^T r_2 & r_2^T r_3 \\ \cdot & \cdot & r_3^T r_3 \end{bmatrix} = I$$

$1 =$, $= 0$

R is an orthogonal matrix

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$SO(3) = \{R \in \mathbb{R}^{3\times3} \mid R^T R = R R^T = I, \det(R) = +1\}$

or $R^T \cdot R = \begin{bmatrix} r_1^T \\ r_2^T \\ r_3^T \end{bmatrix} [r_1 \ r_2 \ r_3] = I$ or $R \cdot R^T = I$

$$\det(R^T R) = \det R^T \cdot \det R = (\det R)^2 = 1, \det R = \pm 1$$

As $\det R = r_1^T (r_2 \times r_3) = 1 \Rightarrow \det R = 1$

$\shortparallel$

r_1

2.2 Rotational Motion in $\mathbb{R}^3$

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Definition:

$$SO(3) = \left\{ R \in \mathbb{R}^{3\times 3} \middle| R^T R = I, \det R = 1 \right\}$$

and

$$SO(n) = \left\{ R \in \mathbb{R}^{n\times n} \middle| R^T R = I, \det R = 1 \right\}$$

"What is a Group?" – Question posed to 7 year old Terry Tao.

$(G, \cdot)$

M.A.C.: What is a group?

T.T.: A set which is mapped onto itself by a binary operation. The binary operation is associative, and the set has an identity e such that $e \times x$ equals x for all x in the set. Also, for each x in the set there is an inverse x' in the set such that $x' * x$ equals e.

M.A.C = M. A. (Ken) Clements
T.T. = Terrence Tao

2.2 Rotational Motion in $\mathbb{R}^3$

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◇ Review: Group

$(G, \cdot)$ is a group if:

1. $g_1, g_2 \in G \Rightarrow g_1 \cdot g_2 \in G$

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2. $\exists!\, e \in G$, s.t. $g \cdot e = e \cdot g = g, \forall g \in G$

2.2 Rotational Motion in $\mathbb{R}^3$

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3. $\forall g \in G, \exists!\, g^{-1} \in G,$ s.t. $g \cdot g^{-1} = g^{-1} \cdot g = e$

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1. $g_1, g_2 \in G \Rightarrow g_1 \cdot g_2 \in G$ closure
2. $\exists!\, e \in G$, s.t. $g \cdot e = e \cdot g = g, \forall g \in G$ identity
3. $\forall g \in G, \exists!\, g^{-1} \in G$, s.t. $g \cdot g^{-1} = g^{-1} \cdot g = e$ inverse
4. $g_1 \cdot (g_2 \cdot g_3) = (g_1 \cdot g_2) \cdot g_3$ associative

♢ Review: Examples of group

1. $(\mathbb{R}^3, +)$

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2. $(\{0, 1\}, + \mod 2)$

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3. $(\mathbb{R}, \times)$ Not a group (Why?)

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5. $S^1 \triangleq \{z \in \mathbb{C} \| z| = 1\}$

Property 1: $SO(3)$ is a group under matrix multiplication.

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Property 1: $SO(3)$ is a group under matrix multiplication.

Proof :

1. If $R_1, R_2 \in SO(3)$, then $R_1 \cdot R_2 \in SO(3)$, because
   - $(R_1 R_2)^T (R_1 R_2) = R_2^T (R_1^T R_1) R_2 = R_2^T R_2 = I$
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   - $\det(R_1 \cdot R_2) = \det(R_1) \cdot \det(R_2) = 1$
2. $e = I_{3\times 3}$

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2. $e = I_{3\times 3}$
3. $R^T \cdot R = I \Rightarrow R^{-1} = R^T$

$R_1(R_2R_3) = (R_1R_2)R_3$

□

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Configuration and rigid transformation:

- $R_{ab} = [x_{ab}\ y_{ab}\ z_{ab}] \in SO(3)$
  Configuration Space

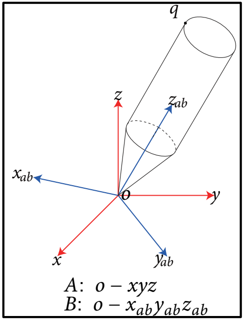

Figure 2.3

2.2 Rotational Motion in $\mathbb{R}^3$

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□ Configuration and rigid transformation:

- $R_{ab} = [x_{ab}\ y_{ab}\ z_{ab}] \in SO(3)$
  Configuration Space

(q in body frame) q_b

(q in spatial frame) q_a

- Let $q_b = \begin{bmatrix} x_b \\ y_b \\ z_b \end{bmatrix} \in \mathbb{R}^3$: coordinates of q in B.

$$q_a = x_{ab} \cdot x_b + y_{ab} \cdot y_b + z_{ab} \cdot z_b$$

$$= [x_{ab}\ y_{ab}\ z_{ab}] \begin{bmatrix} x_b \\ y_b \\ z_b \end{bmatrix} = R_{ab} \cdot q_b$$

R_{ab} transforms vec in {B} to vec {A}

$q_a = R_{ab} q_b$

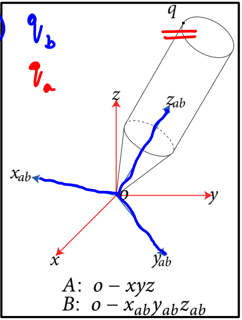

Figure 2.3

2.2 Rotational Motion in $\mathbb{R}^3$

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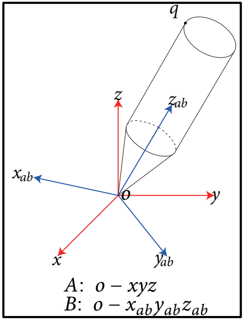

Figure 2.3

- A configuration $R_{ab} \in SO(3)$ is also a transformation:

$$R_{ab} : \mathbb{R}^3 \to \mathbb{R}^3, R_{ab}(q_b) = R_{ab} \cdot q_b = q_a$$

A config. $\Leftrightarrow$ A transformation in $SO(3)$

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Property 2: R_{ab} preserves distance between points and orientation.

1. $\|R_{ab} \cdot (p_b - p_a)\| = \|p_b - p_a\|$
2. $R(v \times \omega) = (Rv) \times R\omega$

Background

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Property 2: R_{ab} preserves distance between points and orientation.

1. $\|R_{ab}\cdot(p_b - p_a)\| = \|p_b - p_a\|$
2. $R(v\times\omega) = (Rv)\times R\omega$

Proof :

For $a \in \mathbb{R}^3$, let $\hat{a} = \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix}$

Note that $\hat{a}\cdot b = a\times b$

1 follows from
$$\|R_{ab}(p_b - p_a)\|^2 = (R_{ab}(p_b - p_a))^T R_{ab}(p_b - p_a) = (p_b - p_a)^T R_{ab}^T R_{ab}(p_b - p_a) = \|p_b - p_a\|^2$$

2 follows from $R\hat{v}R^T = (Rv)^\wedge$ (prove it yourself) □

To show: $R\hat{v}w = \widehat{Rv}\,Rw$

$R\hat{v}R^TRw$

$R\hat{v}w$

Background :

hat map : $\wedge : \mathbb{R}^3 \longrightarrow$ skew symmetric matrix $so(3)$

$A^T = -A$

$$a = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}, \quad \hat{a} = \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix}$$

$$a \times b = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \times \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} -a_3 b_2 + a_2 b_3 \\ a_3 b_1 - a_1 b_3 \\ -a_2 b_1 + a_1 b_2 \end{bmatrix} = \hat{a}\, b$$

Claim : $R \hat{v} R^T = \widehat{Rv}$

Proof : Let $R = \begin{bmatrix} - r_1^T - \\ - r_2^T - \\ - r_3^T - \end{bmatrix} \iff R^T = \begin{bmatrix} | & | & | \\ r_1 & r_2 & r_3 \\ | & | & | \end{bmatrix}$

$$RHS = \widehat{Rv} = \widehat{\begin{bmatrix} - r_1^T - \\ - r_2^T - \\ - r_3^T - \end{bmatrix} v} = \widehat{\begin{bmatrix} r_1^T v \\ r_2^T v \\ r_3^T v \end{bmatrix}}$$

$$LHS = R \hat{v} R^T = R \hat{v} \begin{bmatrix} r_1 & r_2 & r_3 \end{bmatrix}$$

$$= R \begin{bmatrix} \hat{v} r_1 & \hat{v} r_2 & \hat{v} r_3 \end{bmatrix}$$

$$= \begin{bmatrix} r_1^T \\ r_2^T \\ r_3^T \end{bmatrix} \begin{bmatrix} \hat{v} r_1 & \hat{v} r_2 & \hat{v} r_3 \end{bmatrix}$$

$$= \begin{bmatrix} r_1^T \hat{v} r_1 & r_1^T \hat{v} r_2 & r_1^T \hat{v} r_3 \\ r_2^T \hat{v} r_1 & r_2^T \hat{v} r_2 & r_2^T \hat{v} r_3 \\ r_3^T \hat{v} r_1 & r_3^T \hat{v} r_2 & r_3^T \hat{v} r_3 \end{bmatrix} = \widehat{\begin{bmatrix} v^T r_1 \\ v^T r_2 \\ v^T r_3 \end{bmatrix}} = \widehat{\begin{bmatrix} r_1^T v \\ r_2^T v \\ r_3^T v \end{bmatrix}} = \text{RHS}$$

$$r_1 \cdot (\hat{v} r_1) = r_1 \cdot (v \times r_1) = 0$$

$$r_2 \cdot (v \times r_1) = v \cdot (r_1 \times r_2) = v \cdot r_3$$

$$r_1^T \hat{v} r_2 = -v \cdot r_3$$

$$r_2^T \hat{v} r_2 = 0$$

$$r_3^T \hat{v} r_3 = 0$$

$$a \cdot (b \times c) = c \cdot (a \times b) = b \cdot (c \times a)$$

(scalar triple product)

Today

Chapter 2 Rigid Body Motion

1 Rigid Body Transformations

- Length Preserving: $||g(p) - g(q)|| = ||p - q||$
- Orientation Preserving: $g_*(v \times w) = g_*(v) \times g_*(w)$

2 Rotational motion in $\mathbb{R}^3$

- Rotation Matrix
  - Represents *configuration*
  - Represents *(rotational) transformation*
- Rotation Matrices with matrix multiplication form a *Group*
- Rotational Transformation is a *Rigid Body Transformation*