Homework 8

EECS/BioE/MechE C106A/206A
Introduction to Robotics

Due: November 1, 2022

Note: This problem set includes a programming components. Your deliverables for this assignment are:

1. A PDF file submitted to the `HW8 (pdf)` Gradescope assignment with all your work and solutions to the written problems.
2. The provided `hw8.py` file submitted to the `HW8 (code)` Gradescope assignment with your implementation to the programming components.

Problem 1. Dynamics of a Mass-Spring System

Figure 1 shows a system involving a mass m and a spring with spring constant k on an incline. Pick a suitable set of generalized coordinates (you should only need one), and use Lagrangian dynamics to find the equations of motion of the mass-spring system. State the Inertia matrix, Coriolis matrix, and Gravity vector for this system (since this is a one dimensional problem, these will all just be scalars). What is the physical meaning of the generalized forces Υ in this case?

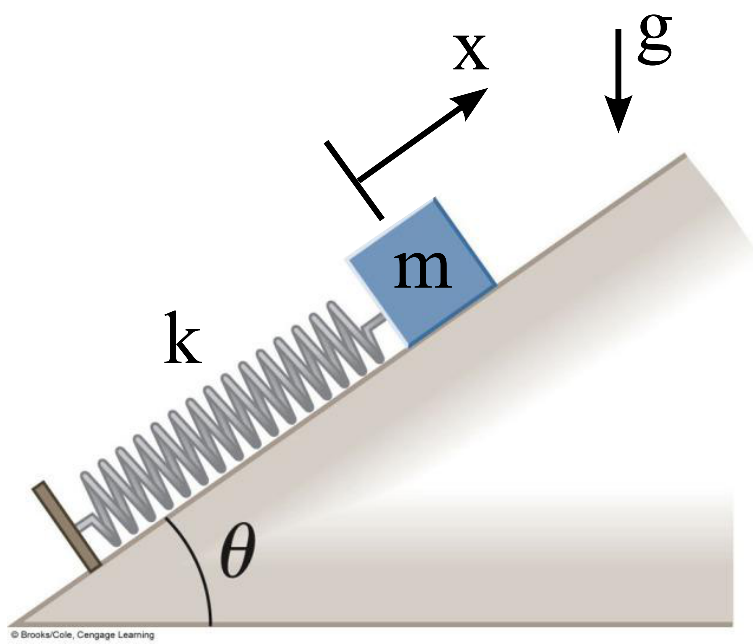


Figure 1: Mass-spring system on a slope.

Problem 2. Dynamics of a Double Pendulum

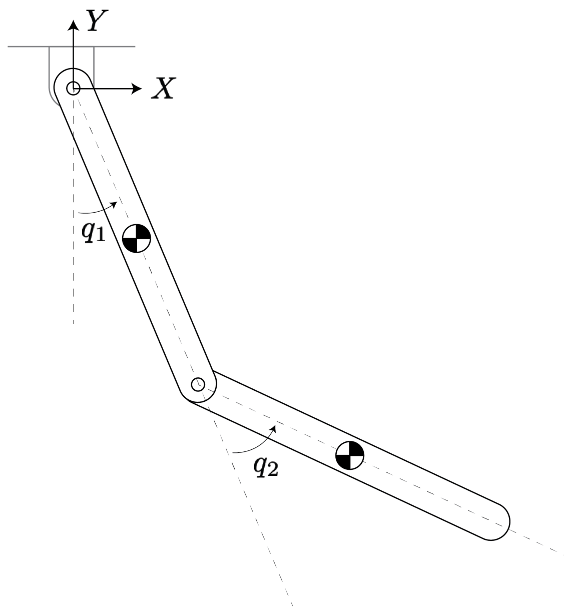


Figure 2: Double pendulum. $z-$axis points directly out of the plane of the paper.

Figure 2 shows a double pendulum hanging from a rigidly fixed pivot. Both arms have length L, mass m, and moments of inertia I about the $z-$axis (pointing out of the plane of the paper) at the center of mass. Both arms have a uniform mass density so that their center of masses are at their centers. Gravity points in the negative y direction.

Pick a suitable set of generalized coordinates (you should need two), and use Lagrangian dynamics to find the equations of motion of the double pendulum. Instead of doing this by hand, you will be making use of the SymPy Python module to save you time, effort, and sanity.

A computer algebra system like SymPy, the MATLAB symbolic toolbox, and Mathematica lets you write code to perform symbolic mathematical computation. This is very useful when you need to perform some tedious algebra and want your final answer to be in terms of some variables. This includes taking integrals, derivatives, matrix exponentials, eigenvalues, etc.

Fill out `hw8.py` to determine the Inertia matrix, Gravity vector, and a possible Coriolis matrix for this system. What is the physical meaning of the generalized forces Υ in this case?

The autograder will look at the variables defined in `hw8.py` to grade you on the correctness of

1. The kinetic energy T of the system.

2. The potential energy V of the system.

3. The necessary derivatives of the Lagrangian: $\frac{\partial L}{\partial q}$ and $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right)$.

4. The inertia matrix $M(q)$ of the system.

5. The Coriolis term $C(q, \dot{q})\dot{q}$ and your choice of Coriolis matrix $C(q, \dot{q})$.

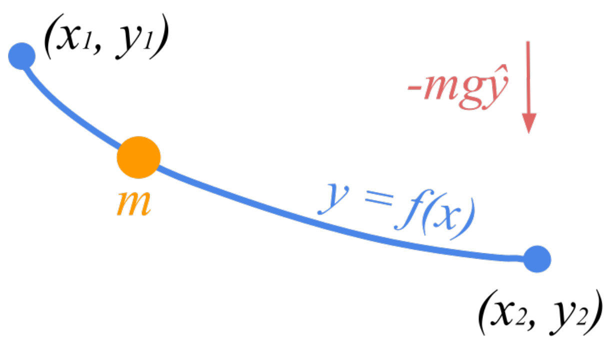
6. The gravity vector $N(q, \dot{q})$.

Problem 3. Bonus: Variations Are the Spice of Life

In 1696, Johann Bernoulli sent out the following announcement to the mathematical community:

> *I, Johann Bernoulli, address the most brilliant mathematicians in the world. Nothing is more attractive to intelligent people than an honest, challenging problem, whose possible solution will bestow fame and remain as a lasting monument. Following the example set by Pascal, Fermat, etc., I hope to gain the gratitude of the whole scientific community by placing before the finest mathematicians of our time a problem which will test their methods and the strength of their intellect. If someone communicates to me the solution of the proposed problem, I shall publicly declare him worthy of praise.*

What problem did Bernoulli so dramatically pose to his peers? Consider the following setup. Suppose you have two points in space, (x_1, y_1) and (x_2, y_2), and a mass that travels on some frictionless path $y = f(x)$ between the two points.



Above: A mass m travels along a path $y = f(x)$ under the influence of gravity

How can we find the path $y = f(x)$ that minimizes the travel time between (x_1, y_1) and (x_2, y_2)?
This famous problem is known as the *brachistochrone problem*, and is something that is rooted firmly in the foundations of Lagrangian mechanics. A popular solution to this problem comes from the field of *calculus of variations*, which has numerous applications in dynamics, path planning, and optimal control theory. Let's tackle this problem and earn the praise of our good friend Bernoulli!

1. Let's begin by describing the motion of the particle in this problem. Assuming that $y_2 = 0$, Argue using conservation of energy that the speed of the particle is computed:

$$\frac{ds}{dt} = v = \sqrt{2gy} \tag{1}$$

   Where g is the gravitational constant, s is the total distance traveled by the particle, and $v = \frac{ds}{dt}$ is the particle's speed. *Hints: Speed is the magnitude of the velocity vector of a particle. The normal force from the wire does no work on the particle.*

2. Using your formula for the velocity of a particle, prove that the time T it takes to get from position 1 to position 2 is computed by the integral:

$$T(y) = \frac{1}{\sqrt{2g}} \int_{x_1}^{x_2} \sqrt{\frac{1 + (y'(x))^2}{y(x)}} dx \tag{2}$$

   Where $y' = \frac{dy}{dx}$. Notice how this time directly depends on the shape of the path between the two points! *Hint: Expand ds in terms of dx and dy - how can you use this to reduce your integral to the form above?*

3. We now face an extremely tough optimization problem! We'd like to find a function $y(x)$ that *minimizes* the total trajectory time between positions 1 and 2.

$$y^*(x) = \underset{y \in \mathcal{C}^1(\mathbb{R})}{\operatorname{argmin}} \frac{1}{\sqrt{2g}} \int_{x_1}^{x_2} \sqrt{\frac{1 + (y'(x))^2}{y(x)}} dx \tag{3}$$

   Here, we look to perform our optimization over the space of *continuous and differentiable functions*, $\mathcal{C}^1(\mathbb{R})$. In a seminal result of the field of calculus of variations, Leonhard Euler proposed the following: a function $g(x, y, y')$ that produces an extrema of the integral:

$$G = \int_a^b g(x, y(x), y'(x) dx \tag{4}$$

   Must satisfy the following condition, known as the Euler-Lagrange equation:

$$\frac{d}{dx}\Big(\frac{\partial g}{\partial y'}\Big) - \frac{\partial g}{\partial y} = 0 \tag{5}$$

   Incredibly, we find that the solution condition of this optimization problem is *exactly* the equation we've been using to solve our Lagrangian mechanics problems!
   This equation would prove quite challenging to work with directly. Let's see if there are any useful facts it implies for our problem. We'll do this in a couple of steps. First, we identify g as the integrand of the expression for $T(y)$ that we previously derived. Write the derivative of $g(x, y, y')$ with respect to x in terms of the following terms:

$$\frac{\partial g}{\partial y},\ \frac{\partial g}{\partial y'},\ y',\ y'' \tag{6}$$

   Where $y' = \frac{dy}{dx}$ and $y'' = \frac{d^2y}{dx^2}$. *Hints: Apply the multivariable chain rule. Is there any explicit dependence of g on x?*

4. Let's now work with the Euler-Lagrange equation. As a first step, let's multiply both sides by y' to get:

$$\frac{d}{dx}\Big(\frac{\partial g}{\partial y'}\Big)y' - \frac{\partial g}{\partial y}y' = 0 \tag{7}$$

Using this as a starting point, prove the following:

$$\frac{\partial g}{\partial y'}y' - g = \text{Constant} \tag{8}$$

Hint: using your chain rule from the previous part, substitute in for the second product in the product of the Euler Lagrange equation with y'. See if you can recognize a product rule anywhere in the result!

5. Let's start breaking down our expression from the previous part. The derivative $\frac{\partial g}{\partial y'}$ is equal to the following:

$$\frac{\partial g}{\partial y'} = \frac{y'}{\sqrt{y(1+y'^2}} \tag{9}$$

After plugging this into the constant equation we derived in the previous part, we may simplify our entire equation down to the following:

$$y(1+y'^2) = \frac{1}{K^2} = C \tag{10}$$

Where K is the constant from the previous part and C is a new constant we define for convenience. Show that this equation implies the following:

$$x = \int \sqrt{\frac{y}{C-y}}dy \tag{11}$$

Hint: recall the definition of y' as the derivative of y with respect to x.

6. Using the trig substitution change of variables:

$$y = C\sin^2(\tfrac{\theta}{2}) \tag{12}$$

Solve the integral above for x in terms of a parametric variable θ. You may use a $+B$ for the integration constant. Once you solve for x as a parametric equation, you'll have a pair of parametric equations:

$$(x(\theta), y(\theta)) \tag{13}$$

That solve the brachistochrone problem. Congratulations!