Homework 7

EECS/BioE/MechE C106A/206A
Introduction to Robotics

Due: October 25, 2022

Problem 1. Jacobian for a 4DOF manipulator

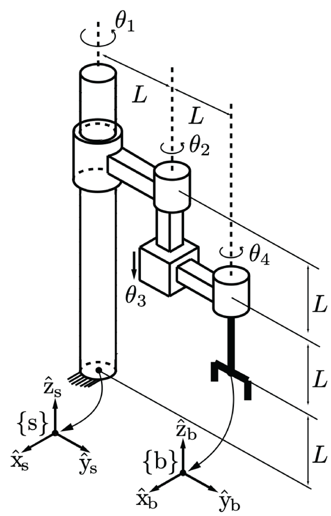

Figure 1: A four degree of freedom manipulator

Figure 1 shows a 4DOF manipulator with 3 revolute joints and 1 prismatic joint (joint 3) in its initial configuration $\theta = 0$.

(a) Compute the spatial Jacobian J^s and the body Jacobian J^b of the manipulator in the configuration shown.

(b) Now let the robot move so that $\theta_2 = \pi/2$, with all other joints remaining at zero. Compute the spatial Jacobian J^s and body Jacobian J^b in the new configuration.

(c) In which configurations, the one in part (a), or the one in part (b), is the robot in a singular configuration? Justify.

(d) During the execution of a smooth joint trajectory $\theta(t) \in \mathbb{R}^4$, the robot passes through the configuration from part(a) with joint velocities $\dot{\theta}(t) = (0, -1/L, 1, 1/L)$. Find the velocity of the origin of the end effector as seen from the spatial frame at that instant. Note that here we are asking for the velocity of the *point* at the origin of the tool frame, so your answer should just be a vector $\dot{p}_{sb} \in \mathbb{R}^3$.

Problem 2. Singularities of Euler Angles

(a) Write down the adjoint of a rotation about the origin by rotation matrix R.

(b) Using this, demonstrate that the singularity of ZYX (extrinsic) Euler angles occurs when $\theta_2 = \frac{\pi}{2}$. Intuitively, why does this occur?

Hint: You should model these Euler angles as the product of three revolute joints along the X, Y, Z axes and show that when the second angle is $\pi/2$ the Jacobian loses rank.

(c) Prove that any rotation represented by three rotations about arbitrary axes ($R = e^{\hat{\omega}_1\theta_1}e^{\hat{\omega}_2\theta_2}e^{\hat{\omega}_3\theta_3}$) will have a singularity.
Hint 1: First prove the case where $a\omega_1 + b\omega_2 = \omega_3$ (such as with ZYZ Euler angles) i.e. when the three axes are linearly dependent. Then prove the harder case where $\omega_1, \omega_2, \omega_3$ are linearly independent.
Hint 2: For the second part, finding a set of angles in terms of $\omega_1, \omega_2, \omega_3$ that will always produce a singular configuration is sufficient. Try drawing it out!

Problem 3. Kinematic Singularity: prismatic joint perpendicular to two parallel coplanar revolute joints

A prismatic joint with twist $\xi_3 = (v_3, 0)$ is normal to a plane containing two parallel revolute axes $\xi_i = (q_i \times \omega_i, \omega_i),\ i = 1, 2$ if

- $v_3^T \omega_i = 0$
- $v_3^T (q_1 - q_2) = 0$
- $\omega_1 = \pm\omega_2$

Show that when this occurs, any six degree of freedom manipulator is at a singular configuration. Give an example of a manipulator exhibiting such a singularity.

Problem 4. Manipulability Agility Ability

In robotics, we're often concerned with the question of how easily our robot arms move in space. We know that in certain configurations, we encounter kinematic *singularities*, in which the robot will no longer be able to enjoy its usual range of motion. How can we measure how close we are to encountering one of these singularities? We may use a *manipulability measure.*

One measure of manipulability is the product of the singular values of the spatial jacobian:

$$\mu(\theta) = \prod_{i=1} \sigma_i(\theta) \tag{1}$$

Where σ_i is the i^{th} *singular value* of the robot's spatial jacobian, $J^s(\theta)$. Note that Π means "take the product" in the same way that Σ means "take the sum."

Before we analyze this special function, we'll need a few facts from linear algebra:

1. The null space of a matrix $A \in \mathbb{R}^{m \times n}$ is the set of all nonzero vectors $v \in \mathbb{R}^n$ such that $Av = 0$.

2. For all matrices $A \in \mathbb{R}^{m \times n}$, the null space of A equals the null space of $A^T A$.

Using these facts, let's examine some properties of the manipulability measure!

(a) The singular values σ_i of a matrix $A \in \mathbb{R}^{m \times n}$ are defined as the square roots of the eigenvalues of $A^T A$:

$$\sigma_i = \sqrt{eig(A^T A)} \tag{2}$$

Prove that if the spatial jacobian $J^s(\theta)$ has a singularity, it will have at least one singular value $\sigma_i = 0$. *Hint: when a matrix is singular, at least one of its eigenvalues is zero.*

(b) Prove that the manipulability measure:

$$\mu(\theta) = \prod_{i=1} \sigma_i(\theta) \tag{3}$$

Is equal to zero whenever the Jacobian has a singularity.

(c) Let's graphically interpret the manipulability function using ZYX Euler angles. Letting $\theta_1 = \theta_3 = 0$, find the spatial jacobian of the ZYX Euler angles as a function of θ_2. Plot the manipulability measure $\mu(\theta_2)$ of this jacobian as a function of the second Euler angle on the domain $\theta_2 \in [0, 2\pi)$. Using your knowledge of Euler angles, interpret the locations of the minima and maxima of $\mu(\theta_2)$.

Note: You may use a computer plotting software such as Python, MATLAB, or Desmos.
Hint: Refer to your solution to question 2.