Homework 6

EECS/BioE/MechE C106A/206A
Introduction to Robotics

Due: October 18, 2022

Theory

Velocities

We have spoken about rigid body motion so far as a transformation between two points. The g matrix relates the positions of two coordinate frames to one another. Often, we also want to talk about the velocity, or the rate of change of the position of some point with respect to a given reference frame.

When dealing with robots, there are two reference frames that are relevant: the spatial and the body frames.

- Let's fix frame A and consider it our spatial frame (this might be our world frame).
- Let's say frame B is moving - we'll call this our body frame (this might be one of our robot arms).
- Finally, let's say we have some point q that's attached to B (this might be the tip of a gripper attached to the arm).

Let's try to find the velocity of the arbitrary point q. Your intuition should tell you that the velocity of the point q with respect to frame B is just 0. Since q and B are moving together, they have no velocity with respect to each other. There are, however, other ways in which we can express the motion of q.

Spatial Velocity

Because point q is moving, its location is time-dependent.

$$q_a(t) = g_{ab}(t)q_b$$

The velocity of this point with respect to frame A and expressed in terms of A is the spatial velocity.

$$v_{q_a}(t) = \hat{V}^s_{ab} q_a = \dot{g}_{ab}(t) g^{-1}_{ab}(t) q_a$$

$$\hat{V}^s_{ab} = \dot{g}_{ab}(t)g^{-1}_{ab}(t)$$

It turns out that $\hat{V}^s_{ab}$ is a skew-symmetric matrix. We can apply the vee operator to get the spatial velocity, which can be expressed as a twist:

$$(\hat{V}^s_{ab})^\vee = V^s_{ab} = \begin{bmatrix} v^s_{ab} \\ \omega^s_{ab} \end{bmatrix}$$

Body Velocity

Say we'd also like to express our velocity with respect to the body frame. Your intuition should tell you that the velocity of the point q with respect to frame B is just 0: since q and B are moving together, they have no velocity with respect to each other. Thus, the velocity of q with respect to frame B is not how we define the body velocity. Instead, we say that the body velocity is the velocity of the point q relative to A, but now expressed in terms of B.

$$v_{q_b}(t) = \hat{V}^b_{ab}q_b = g^{-1}_{ab}(t)\dot{g}_{ab}(t)q_b$$

$$\hat{V}^b_{ab} = g^{-1}_{ab}(t)\dot{g}_{ab}(t)$$

This is also a skew-symmetric matrix and can be expressed as a twist:

$$(\hat{V}^b_{ab})^\vee = V^b_{ab} = \begin{bmatrix} v^b_{ab} \\ \omega^b_{ab} \end{bmatrix}$$

Another Interpretation of the Velocities

Each part of the spatial and body velocity twists can be found by interpreting the motion of our robot in the following way:

- v^s_{ab}: Velocity of a point attached to B traveling through the origin of A, expressed in terms of A
- ω^s_{ab}: Angular velocity of B wrt A, expressed in terms of A
- v^b_{ab}: Velocity of the origin of B wrt A, expressed in terms of B
- ω^b_{ab}: Angular velocity of B wrt A, expressed in terms of B

Adjoints

The Adjoint transformation associated with $g \in SE(3)$ is a 6×6 matrix Ad_g and is defined as follows:

$$Ad_g = \begin{bmatrix} R & \hat{p}R \\ 0 & R \end{bmatrix}$$

The adjoint transformation is invertible and is used to map twists from one coordinate frame to another: $\xi' = Ad_g\xi$

The inverse of an adjoint is

$$Ad_g^{-1} = \begin{bmatrix} R^T & -R^T\hat{p} \\ 0 & R^T \end{bmatrix}$$

In the context of motion, it can map body velocity twist coordinates to spatial velocity twist coordinates. As a result, we have

$$V_{ab}^s = Ad_{g_{ab}} V_{ab}^b$$

Further recall that for any $\xi \in \mathbb{R}^6$, we have

$$Ad_g \xi = (g \cdot \hat{\xi} \cdot g^{-1})^\vee$$

To compose velocities between coordinate transformations, we need to take advantage of this idea

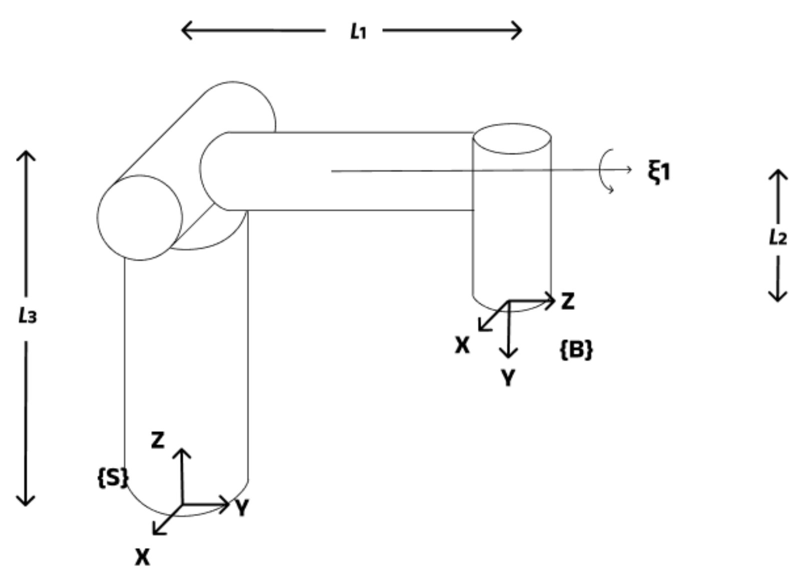
$$V_{ac}^s = V_{ab}^s + Ad_{g_{ab}} V_{bc}^s$$
$$V_{ac}^b = Ad_{g_{bc}^{-1}} V_{ab}^b + V_{bc}^b$$

For a full proof of this, please consult MLS Propositions 2.14 and 2.15 (on page 59).

Problems

Problem 1. Velocities Practice

Find the spatial and body velocity twists for the fixed frame S and the moving frame B in terms of θ, $\dot{\theta}$, and the lengths shown in the diagram. Assume we only consider movement about the ξ_1 axis.



Problem 2. Twists as Velocities

Recall that Chasle's theorem allows us to express any rigid body transform $g \in SE(3)$ as the exponential of a (not necessarily unit) twist $\xi \in \mathfrak{se}(3)$ as $g = e^{\hat{\xi}}$. Consider a trajectory $g(t) \in SE(3)$ that evolves from an initial configuration $g(0)$ to a final configuration $g(1)$ according to the following formula:

$$g(t) = e^{\hat{\xi}t} \cdot g(0) \tag{1}$$

where $t \in [0, 1]$ denotes time. $g(t)$ evolves according to a *constant screw motion*. In this problem, we are going to interpret a twist ξ as a velocity.

(a) Given a desired initial configuration $g_0 \in SE(3)$ and a desired final configuration $g_1 \in SE(3)$, how might we find a twist $\xi \in \mathfrak{se}(3)$ so that a smooth trajectory of the form (1) takes us from $g(0) = g_0$ to $g(1) = g_1$? You do not need to explicitly solve for the twist.

(b) For $t \in (0, 1)$, find the spatial rigid body velocity $V^s(t)$ of the trajectory you constructed in part (a). Does this velocity depend on time?

(c) For $t \in (0, 1)$, find the body velocity $V^b(t)$ of the trajectory you constructed in part (a). Does this velocity depend on time?

(d) Let a $g \in SE(3)$ be given, with exponential coordinates ξ (a not necessarily unit twist) so that $g = e^{\hat{\xi}}$. Interpret the twist ξ as a rigid body velocity that, when performed uniformly for 1 second, brings a rigid body from the identity configuration to the configuration g. In this way, interpret twists (and the idea of exponential coordinates) in terms of rigid body velocities.

Problem 3. Velocities as Twists

Consider a smooth rotational trajectory $R(t) \in SO(3)$ where $t \in [0, \infty)$ denotes time. In this problem, we will derive the notion of angular and rigid body velocities directly from our knowledge of exponential coordinates.

(a) Let $t \in [0, \infty)$ and a small $\Delta t > 0$ be given. Argue that there exists $\hat{\omega} \in \mathfrak{so}(3)$ such that

$$R(t + \Delta t) = e^{\hat{\omega}\Delta t} \cdot R(t) \tag{2}$$

Note that ω is a function of both t and Δt.

(b) Now take the limit as $\Delta t \to 0$. Show that in this limit, $\hat{\omega}$ approaches $\dot{R}R^T = \hat{\omega}^s(t)$. That is, in the limit, this infinitessimal rotation approaches the spatial angular velocity of R. *Hint: It may help to recall that for small Δt we have $e^{A\Delta t} \approx I + A\Delta t$. It may also help to recall the limit definition of the derivative.*

(c) Conclude that the spatial angular velocity of R is simply the *instantaneous rotation axis* of the body, with magnitude equal to the instantaneous angular speed.

(d) Repeat the exercise in parts (a)-(c) except with a smooth *rigid-body motion* trajectory $g(t) \in SE(3)$. Interpret the spatial velocity $V^s(t)$ in terms of the twist associated with the *instantaneous screw motion* that the body is undergoing at time t.

Problem 4. Properties of the Adjoint

(a) Show that $(\mathrm{Ad}_g)^{-1} = \mathrm{Ad}_{g^{-1}}$ for all $g \in SE(3)$.

(b) Show that $\mathrm{Ad}_{g_1 g_2} = \mathrm{Ad}_{g_1} \mathrm{Ad}_{g_2}$ for all $g_1, g_2 \in SE(3)$.

(c) Prove MLS Proposition 2.15: $V^b_{ac} = \mathrm{Ad}_{g^{-1}_{bc}} V^b_{ab} + V^b_{bc}$.

Hint: It may help to take the "hat" of both sides first.