Homework 1

EECS/BioE/MechE C106A/206A
Introduction to Robotics

Due: September 6, 2022

Note: Problems marked [bonus] will be eligible for a (very) small amount of extra credit, though you cannot receieve more than a full score on the homework as a whole. We encourage you not to spend exorbitant amounts of time on these questions, and as such, you may receieve partial credit for attempting them.

Note 2: This problem set includes a programming component. Your deliverables for this assignment are:

1. A PDF file submitted to the `HW1 (pdf)` Gradescope assignment with all your work and solutions to the written problems.

2. The provided `hw1.py` file submitted to the `HW1 (code)` Gradescope assignment with your implementation to the programming components.

Problem 1. Properties of Rotations

State whether each transformation matrix below is a valid rotation. Justify.

(a) $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

(b) $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$

(c) $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

(d) $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

Problem 2. Euler Angles

Consider two initially coincident reference frames, A and B. Frame B is then rotated about the Z axis by $\pi/4$ radians.

a) Sketch the coordinate frames A and B after the rotation.

b) Write the rotation matrix R_{AB} that will take a point from the B frame and represent it in the A frame.

c) Write the rotation matrix R_{BA}.

d) What are the coordinates in frame A of a point with coordinates $p_B = [0, 0, 1]^T$ given with respect to frame B?

e) What are the coordinates in frame B of a point with coordinates $p_A = [1, 1, 0]^T$ given with respect to frame A?

Problem 3. Multiple Euler Angles

(a) A frame is rotated first about the Z axis by angle $\frac{\pi}{2}$, then about the mobile Y axis by an angle of $\frac{\pi}{2}$, then about the mobile X axis an angle of $\frac{\pi}{2}$.

(i) Draw the frame before and after the rotation. Label all axes.

(ii) Write the net rotation matrix.

(b) A frame is rotated first about the Z axis by angle $\frac{\pi}{2}$, then about the original Y axis by an angle of $\frac{\pi}{2}$, then about the original X axis an angle of $\frac{\pi}{2}$.

(i) Draw the frame before and after the rotation. Label all axes.

(ii) Write the net rotation matrix.

Problem 4. Rotation Matrices in Action

Valmik wants to plot the motion of a car moving in 2D space for his next research paper, but there's one problem! His plotting code doesn't take the car's orientation into consideration when plotting, so nothing looks right (figure 1).

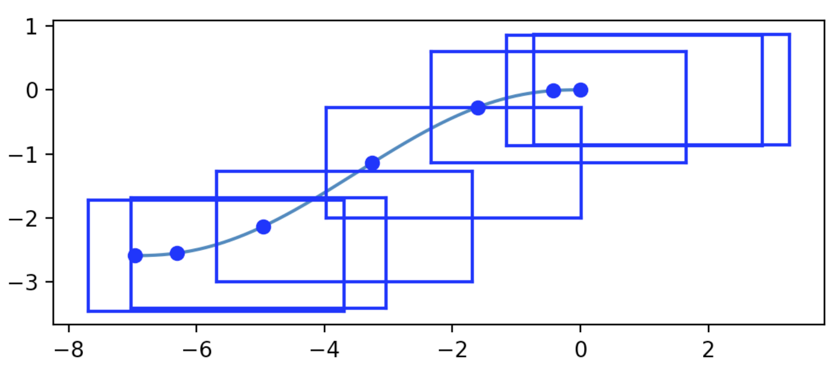

Figure 1: Cars can't move like this!

Edit the `hw1.py` to take orientation into account when plotting the car, and create a visualization by running `car_vis.py`. Once you are satisfied with your visualization, submit only `hw1.py` to the `HW1 (code)` assignment on Gradescope.

[Bonus] Problem 5. Cosine Direction Matrices

The goal of this problem is to make geometric sense of the fact that for any two frames A and B, we have $R_{AB} = R_{BA}^T$. We will do this by finding a geometric interpretation for the entries of the rotation matrix R_{AB}.

First, some notation. For a given matrix R, denote by R_{ij} the entry of the matrix in row i and column j. Let A and B be two reference frames in 3D space with unit axes $\{a_1, a_2, a_3\}$ and $\{b_1, b_2, b_3\}$ respectively. For any two vectors $v, w \in \mathbb{R}^3$ denote by $\angle(v, w) \in [0, \pi]$ the angle between v and w.

Let $R = R_{AB}$ be the rotation matrix that transforms points written in frame B to their coordinates in frame A. We will assume nothing about this matrix other than the fact that it performs this transformation. Note that this means you cannot use the fact that $R^{-1} = R^T$ anywhere in your solutions; indeed, proving this is the point of this problem.

(a) Write down the coordinates of b_i as seen from frame B. Write down an expression for its coordinates as seen from frame A, using R.

(b) Show that

$$R_{ij} = \cos\left(\angle(a_i, b_j)\right) \tag{1}$$

In other words, show that each entry of R_{AB} is simply the cosine of the angle formed between the two corresponding axes of frames A and B.

Hint 1: For two unit vectors in 3D space, we have $\cos\angle(u, v) = u^T v$.

Hint 2: In order to use $\cos\angle(u, v) = u^T v$*, it is imperative that* u *and* v *both be written in the same reference frame!*

(c) Conclude that $R_{AB} = R_{BA}^T$.