Homework 0: Linear Algebra Review

EECS/BioE/MechE C106A/206A
Introduction to Robotics

Due: August 30, 2022

Note: Problems marked [bonus] will be eligible for a (very) small amount of extra credit, though you cannot receieve more than a full score on the homework as a whole. We encourage you not to spend exorbitant amounts of time on these questions, and as such, you may receieve partial credit for attempting them.

Problem 0. Syllabus Quiz

Complete the `Syllabus Quiz` assignment on Gradescope. This will be part of your grade for Homework 0.

Problem 0.5. It's Dangerous to Go Alone

This class can be really rough at times, so it always helps to have someone you can rely on for notes or other assistance when needed. Plus, you will need a partner for the lab, and a few for the final project! For this question, find a study buddy (or buddies) and exchange contact information. Then, state the name and email of your buddy in response to this question. For a chance to score extra credit, attach an epic selfie of you with your buddy as well (a zoom meeting screenshot would also suffice).
Note: Feel free to buddy with as many people as you want, and if someone wants to be your buddy after you already submit this assignment, you don't have to resubmit.

Problem 1. Orthogonal Matrices

Let $\mathbf{R}$ be an $n \times n$ matrix, and let $r_i \in \mathbb{R}^n$ be the i-th column of $\mathbf{R}$. $\mathbf{R}$ is said to be *orthogonal*, if for any $i \neq j$, the vectors r_i and r_j are orthogonal to each other, and each r_i is unit length. We then also say that the vectors $\{r_1, ..., r_n\}$ form an *orthonormal basis* for $\mathbb{R}^n$.

(a) Show that a square matrix $\mathbf{A}$ is orthogonal if and only if $\mathbf{A}^T\mathbf{A} = \mathbf{I}$. *Hint: Consider writing the* (i, j)*th entry of* $\mathbf{A}^T\mathbf{A}$ *in terms of dot products of the columns of* $\mathbf{A}$.

(b) Let $\mathbf{R}$ be an orthogonal $n \times n$ matrix and let u be an $n-$dimensional vector. Show that $||\mathbf{R}u|| = ||u||$. In other words, show that $\mathbf{R}$ preserves norms when it acts on vectors. *Hint: Use the fact that for the standard euclidean norm,* $||u||^2 = u^T u$.

(c) Show that if $\mathbf{R}$ is an orthogonal matrix, then $\det(\mathbf{R}) = \pm 1$. *Hint: Take the determinant on both sides of the equation* $\mathbf{R}^T\mathbf{R} = \mathbf{I}$.

Problem 2. A Shift in Perspective

One linear algebra concept we heavily use is the *change of coordinates*. We will spend some time developing a mathematical framework for describing how rigid objects move relative to each other, the essence of rigid body motion. This all starts from the ideas of a *basis*, a set of vectors which define a coordinate system. While the coordinate transforms introduced in this course on may feel different than the change of basis you may have seen in earlier classes, the math is essentially the same. Consider this problem a refresher on basis concepts and a taste of what's to come.

(a) Given vector v defined in terms of the standard basis and a set of basis vectors $\beta = \{\beta_1, \ldots, \beta_n\}$, compute v_β, vector v in terms of the basis β.

(b) Define $g_{\alpha\beta}$ to be a change of basis matrix from basis β to basis α. Given $g_{\alpha\beta}$, $g_{\gamma\beta}$ and vector v_α, compute v_γ.

(c) True or False: Orthogonality between vectors is independent of choice of basis for those vectors. If true, provide a proof. If false, provide a counterexample.

(d) True or False: For any linearly independent set of vectors, we can pick a basis for those vectors which makes the set orthonormal. If true, provide a proof. If false, provide a counterexample.

Problem 3. The Matrix Exponential: Algebraic Properties

Recall that for a scalar $a \in \mathbb{R}$, we can write its exponential e^a as a Taylor series that converges for any a:

$$e^a = \sum_{n=0}^{\infty} \frac{a^n}{n!} = 1 + a + \frac{a^2}{2!} + \cdots \tag{1}$$

We can similarly use an infinite series to *define* the exponential of a square real $n \times n$ *matrix* $\mathbf{A}$:

$$e^{\mathbf{A}} = \sum_{n=0}^{\infty} \frac{\mathbf{A}^n}{n!} = \mathbf{I} + \mathbf{A} + \frac{1}{2!}\mathbf{A}^2 + \cdots \tag{2}$$

where by convention we take $\mathbf{A}^0$ to be the identity matrix for any square matrix $\mathbf{A}$. The result is also an $n \times n$ matrix. As it turns out, this infinite series converges absolutely for every matrix $\mathbf{A}$. So we use this series to define the *matrix exponential function* $e^{\mathbf{A}}$.

The matrix exponential shows up all over the place in the study of rigid body motion and dynamical systems, especially in the solutions to vector differential equations, as we shall see. We will make heavy use of the matrix exponential in this class. In this problem, you will use the infinite series representation in equation (2) to derive some of the fundamental algebraic properties of this function which will prove very useful in our study of rigid body kinematics.

(a) Show that $e^{\mathbf{0}} = \mathbf{I}$. i.e. the exponential of the zero matrix is the identity matrix.

(b) Show that $(e^{\mathbf{A}})^T = e^{(\mathbf{A}^T)}$.

(c) Let g be any invertible square matrix of the same size as $\mathbf{A}$. Show that $e^{g\mathbf{A}g^{-1}} = ge^{\mathbf{A}}g^{-1}$.
Hint: Start by showing that for all n, $(g\mathbf{A}g^{-1})^n = g\mathbf{A}^n g^{-1}$.

(d) Show that if λ is an eigenvalue of $\mathbf{A}$ then e^{λ} is an eigenvalue of $e^{\mathbf{A}}$.

Hint: Use the series expansion. Show that if v is an eigenvector of $\mathbf{A}$ with eigenvalue λ then it is also an eigenvector of $e^{\mathbf{A}}$ with eigenvalue e^{λ}. i.e. show that $e^{\mathbf{A}}v = e^{\lambda}v$.

Remark: In fact, a suitable converse of the above statement is also true, though more difficult to prove. We can conclude that if the eigenvalues of $\mathbf{A}$ (possibly repeated) are $\lambda_1, ..., \lambda_n$ then the eigenvalues of $e^{\mathbf{A}}$ are exactly $e^{\lambda_1}, ..., e^{\lambda_n}$.

(e) [**Bonus**] Using the previous part, show that $\det(e^{\mathbf{A}}) = e^{\operatorname{tr}\mathbf{A}}$. Conclude that the exponential of any matrix is always invertible.

Hint: What is the relationship between the eigenvalues of a matrix, its determinant and its trace? Also use the remark from the previous part.

Remark: In fact, the inverse of $e^{\mathbf{A}}$ is simply $e^{-\mathbf{A}}$.

Problem 4. Enter the Matrix (Differential Equation)

In this problem, we'll review the solution to an important class of ordinary differential equations. In next week's lecture, we'll see the importance of these equations in describing *rotations of rigid bodies.*

(a) Solve the ordinary differential equation $\frac{dx}{dt} = ax(t)$, for $t \geq 0$, $a \in \mathbb{R}$, assuming the initial condition $x(0) = x_0$. You may assume $x(t) > 0$ for all $t \in \mathbb{R}$.

(b) Find the general solution to the following homogeneous system of linear differential equations:

$$\frac{dx}{dt} = \begin{bmatrix} 2 & 2 \\ 1 & 3 \end{bmatrix} x(t) \tag{3}$$

Your solution should be of the form x(t) $= c_1 e^{\lambda_1 t} x_1 + c_2 e^{\lambda_2 t} x_2$, for $x_1, x_2 \in \mathbb{R}^2$. *Hint: How can diagonalizing the matrix help us find a solution?*

Problem 5. Bonus: Power Trip - Convergence of the Exponential

In this question, we'll prove the challenging yet important fact that for all $A \in \mathbb{R}^{n \times n}$, e^A converges. To prove this, we'll need a few important facts about the convergence of series:

1. If $\sum_{i=1}^{\infty} ||A_i||$ converges, where $A_i \in \mathbb{R}^{n \times n}$ for all i and $|| \cdot ||$ is a valid matrix norm, then $\sum_{i=1}^{\infty} A_i$ is said to *converge normally.*

2. (Comparison test) if $\sum_{i=1}^{\infty} b_i$ converges, and $0 \leq a_i \leq b_i$ for all i, then $\sum_{i=1}^{\infty} a_i$ converges.

Let's begin!

(a) To prove the convergence of the matrix exponential, it will help us to have a *submultiplicative* matrix norm. A submultiplicative norm is defined as follows: The norm $||\cdot||$ is submultiplicative if for all $A, B \in \mathbb{R}^{n\times n}$, $||AB|| \leq ||A|| \cdot ||B||$.
Prove that the Frobenius matrix norm, defined:

$$||A||_F = \sqrt{\text{Trace}(AA^T)} \tag{4}$$

is submultiplicative.

(b) Prove that the matrix exponential, defined:

$$e^A = \sum_{k=0}^{\infty} \frac{A^k}{k!} \tag{5}$$

converges normally for all matrices $A \in \mathbb{R}^{n\times n}$. *Hint: use fact (1) from above together with your submultiplicative norm from part (a).*

Problem 6.

Being able to program in Python is essential for completing lab and homework assignments. Every homework assignment will have some coding portion, and for this homework there's a Jupyter notebook `hw0.ipynb` which serves as a Python crash course.

After completing the notebook, there will be a Python file `rodrigues.py` for you to fill out. Submit this file on gradescope to the assignment titled "HW0 Code".