C106A / 206A - Discussion 9

Friday, October 28, 2022 3:10 PM

Lagrangian Dynamics

↳ Twists, FK, IK, Velocities / Jacobians

↳ Newtonian Mechanics → requires reference frame

$F = mg$ $\quad \sum F = 0$

↳ inertial ref. frames

↳ conservation of energy

Lagrangian Formulation:

1) Write minimal state representation, q

$q = \begin{bmatrix} x(t) \\ a(t) \end{bmatrix}$ not minimal

$q = x$

2) Write kinetic energy, T

3) Write potential energy, V

4) Lagrangian $L := T - V$

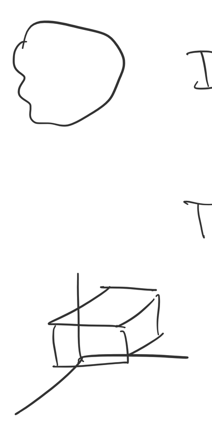
5) Euler-Lagrange: $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = \Upsilon$ "generalized" external forces, "upsilon"

6) $\underbrace{M(q)}_{\text{mass/inertial}}\ddot{q} + \underbrace{C(q,\dot{q})}_{\text{coriolis}}\dot{q} + \underbrace{N(q)}_{\text{gravity vector}} = \Upsilon$

- Common Sources of Energy

↳ Kinetic:
- Linear KE, $T = \frac{1}{2}mv^2$
- Rotational KE, $T = \frac{1}{2}I\dot{\theta}^2$ ↳ inertia

3D



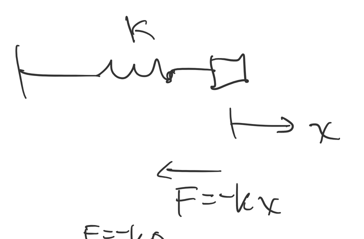
$I_c = \begin{bmatrix} I_{xx} & I_{xy} & I_{xz} \\ " & I_{yy} & I_{yz} \\ " & " & I_{zz} \end{bmatrix}$

$T = \frac{1}{2}\omega^T I_c \omega$

$I_c = \begin{bmatrix} I_{xx} & 0 & 0 \\ 0 & I_{yy} & 0 \\ 0 & 0 & I_{zz} \end{bmatrix}$

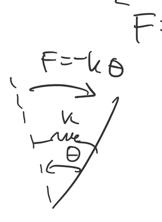
↳ Potential:
- Gravitational PE, $V = mgh$
- Elastic PE,

Linear:



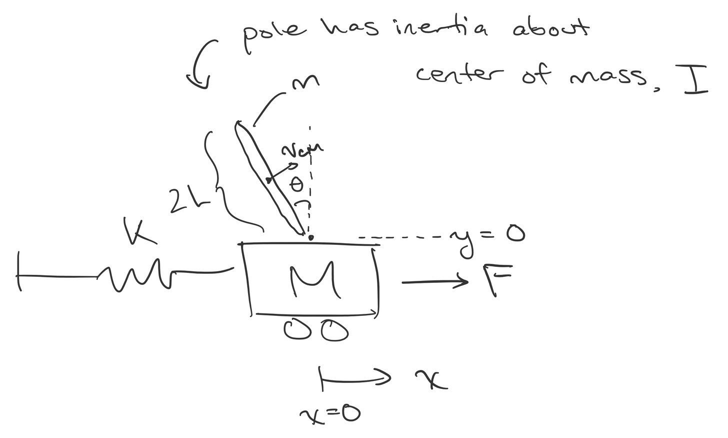
$V = \frac{1}{2}kx^2$

Torsional:



$V = \frac{1}{2}k\theta^2$

3.4)



1) Specify state $q = \begin{bmatrix} x \\ \theta \end{bmatrix}$

2) kinetic energy, $T = \frac{1}{2}M\dot{x}^2 + \underbrace{\frac{1}{2}I\dot{\theta}^2}_{\text{rotational}} + \underbrace{\frac{1}{2}m v_{CM}^2}_{\text{linear}}$

$|v|^2 = v_x^2 + v_y^2$

$\sin(\theta) = \frac{x_r}{L}$

$v_{CM}^2 = v_{CM_x}^2 + v_{CM_y}^2$

$x_{CM} = x - L\sin(\theta)$

$y_{CM} = L\cos(\theta)$

$v_{CM_x} = \dot{x} - L\cos(\theta)\dot{\theta}$ ←

$v_{CM_y} = -L\sin(\theta)\dot{\theta}$ ←

$v_{CM}^2 = \dot{x}^2 - 2L\cos(\theta)\dot{x}\dot{\theta} + L^2\dot{\theta}^2$

3) Potential Energy

$V = \frac{1}{2}kx^2 + mgL\cos(\theta)$

4) $L := T - V = \frac{1}{2}M\dot{x}^2 + \frac{1}{2}I\dot{\theta}^2 + \frac{1}{2}m(\dot{x}^2 - 2L\cos(\theta)\dot{x}\dot{\theta} + L^2\dot{\theta}^2)$
$- \frac{1}{2}kx^2 - mgL\cos(\theta)$

5) $\frac{\partial L}{\partial q} = \begin{bmatrix} \frac{\partial L}{\partial x} \\ \frac{\partial L}{\partial \theta} \end{bmatrix} = \begin{bmatrix} -kx \\ mgL\sin(\theta) + mL\sin(\theta)\dot{x}\dot{\theta} \end{bmatrix}$ ←

$\frac{\partial L}{\partial \dot{q}} = \begin{bmatrix} \frac{\partial L}{\partial \dot{x}} \\ \frac{\partial L}{\partial \dot{\theta}} \end{bmatrix} = \begin{bmatrix} (M+m)\dot{x} - mL\cos(\theta)\dot{\theta} \\ I\dot{\theta} - mL\cos(\theta)\dot{x} + mL^2\dot{\theta} \end{bmatrix}$

$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) = \begin{bmatrix} (M+m)\ddot{x} - mL\cos(\theta)\ddot{\theta} + mL\sin(\theta)\dot{\theta}^2 \\ I\ddot{\theta} - mL\cos(\theta)\ddot{x} + mL\sin(\theta)\dot{x}\dot{\theta} + mL^2\ddot{\theta} \end{bmatrix}$ ←

$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = \begin{bmatrix} (M+m)\ddot{x} - mL\cos(\theta)\ddot{\theta} + mL\sin(\theta)\dot{\theta}^2 + kx \\ I\ddot{\theta} - mL\cos(\theta)\ddot{x} + \cancel{mL\sin(\theta)\dot{x}\dot{\theta}} + mL^2\ddot{\theta} - mgL\sin(\theta) - \cancel{mL\sin(\theta)\dot{x}\dot{\theta}} \end{bmatrix} = \Upsilon = \begin{bmatrix} F \\ \tau \end{bmatrix}$

6) $\underbrace{\begin{bmatrix} M+m & -mL\cos(\theta) \\ -mL\cos(\theta) & I+mL^2 \end{bmatrix}}_{M(q)} \ddot{q} + \underbrace{\begin{bmatrix} 0 & mL\sin(\theta)\dot{\theta} \\ 0 & 0 \end{bmatrix}}_{C(q,\dot{q})} \dot{q} + \underbrace{\begin{bmatrix} kx \\ -mgL\sin(\theta) \end{bmatrix}}_{N(q)} = \Upsilon$

$\ddot{q} = \begin{bmatrix} \ddot{x} \\ \ddot{\theta} \end{bmatrix}$, $\quad \dot{q} = \begin{bmatrix} \dot{x} \\ \dot{\theta} \end{bmatrix}$