EE106A Discussion 11: Midterm Preparation

Problem 1: Using knowledge of adjoints, prove that the forward kinematics map is the same as composing subsequent rigid body transformations.

* Avoid using adjoints w/ FK

why do we not need the updated joint positions?

$$g_{st}(\theta_1) = e^{\hat{\xi}_1 \theta_1} g_{st}(0)$$

$\hookrightarrow$ θ_1 moves

θ_2 fixed

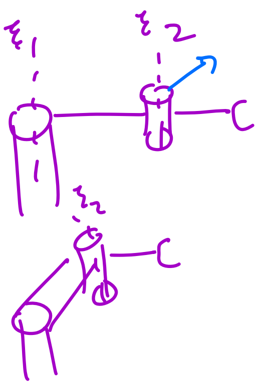
$$\xi_2' = Ad_{e^{\hat{\xi}_1 \theta_1}} \xi_2 \rightarrow \text{new location of } \xi_2$$

Composing transforms: $\underbrace{g_{21}}_{\text{Result of transforming with } \xi_2} \underbrace{g_{10}}_{} \cdot g(0)$ $\rightarrow$ Transforming w/ ξ_1

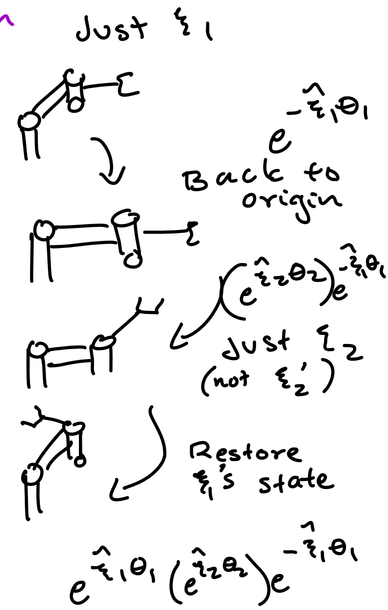
$e^{\hat{\xi}_2 \theta_2}$ $\rightarrow$ Rotate ξ_1 back to the origin

Rotate ξ_2 (avoid using ξ_2')

Rotate back



$$= \underbrace{e^{\hat{\xi}_1 \theta_1}}_{\text{Apply } \xi_1 \text{ again}} \underbrace{\left(e^{\hat{\xi}_2 \theta_2}\right) e^{-\hat{\xi}_1 \theta_1}}_{\text{Kept } \xi_1 \text{ in place Rotated about } \xi_2}$$



$$g_{st}(\theta_1, \theta_2) = \underbrace{e^{\hat{\xi}_2' \theta_2}}_{} e^{\hat{\xi}_1 \theta_1} g_{st}(0)$$

Gets us to end effector position

$$= \left[e^{\hat{\xi}_1 \theta_1} \left(e^{\hat{\xi}_2 \theta_2} \right) \cancel{e^{-\hat{\xi}_1 \theta_1}} \right] \cancel{e^{\hat{\xi}_1 \theta_1}} \cdot g_{st}(0)$$

$$= e^{\hat{\xi}_1 \theta_1} e^{\hat{\xi}_2 \theta_2} g_{st}(0)$$

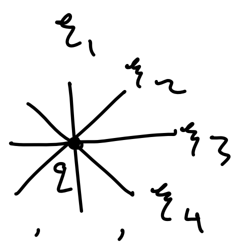
FK map

Problem 2: Prove that the spatial velocity of a manipulator does not depend on the location of the tool frame (as long as it moves with the end-effector).

* See Practice Midterm

Problem 5

Problem 3: Show that a manipulator with 4 intersecting revolute joints ~~at the origin~~ will necessarily have a singularity.



$$J^s = \begin{bmatrix} -\omega_1 \times q & -\omega_2' \times q' & -\omega_3' \times q' & -\omega_4' \times q' \\ \omega_1 & \omega_2' & \omega_3' & \omega_4' \end{bmatrix}$$

↓

Ignore the v components - doesn't matter where reference frame is defined, assume @ origin

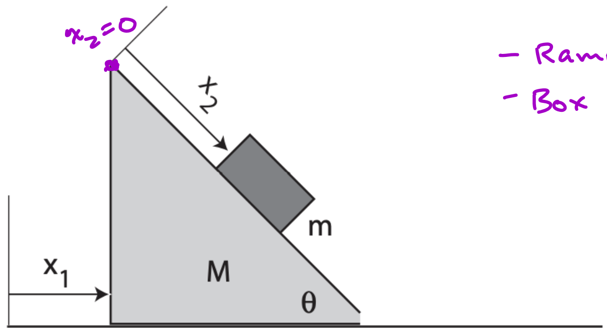
$$J^s = \begin{bmatrix} 0 & 0 & 0 & 0 \\ \omega_1 & \omega_2' & \omega_3' & \omega_4' \end{bmatrix} \Big\} 3 \times 4$$

Max rank of 3 w/ omegas

But we have 4 - can't be linearly independent

↳ Jacobian cannot be rank 4

Problem 4: Calculate the dynamics of the a box with mass m sliding down a ramp of mass M. Assume there is no friction in the system. Then, develop a control law for this motion.



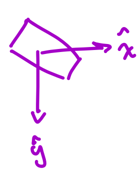
- Ramp moving $(\dot{x}_1)$
- Box sliding $(\dot{x}_2)$

Figure 1: Image sourced from http://www.dzre.com/alex/P441/lectures/lec_18.pdf

1) $q = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

2) KE: Ramp $\rightarrow \frac{1}{2} M \cdot \dot{x}_1^2$

Object $\rightarrow$ Velocities come from $\dot{x}_1$ and $\dot{x}_2$
Want to make sure we account for everything but have no overlap



$$v_{\hat{x}} = \dot{x}_1 + x_2 \cos\theta$$

$$v_{\hat{y}} = \dot{x}_2 \sin\theta$$

$$T_m = \frac{1}{2} m (\dot{x}_1 + \dot{x}_2 \cos\theta)^2 + \frac{1}{2} m (\dot{x}_2 \sin\theta)^2$$

$$= \frac{1}{2} m \dot{x}_1^2 + \frac{1}{2} m (2 \dot{x}_1 \dot{x}_2 \cos\theta) + \frac{1}{2} m \dot{x}_2 \cos^2\theta + \frac{1}{2} m \dot{x}_2 \sin^2\theta$$

$$\frac{1}{2} m \dot{x}_2^2$$

$$= \boxed{\frac{1}{2} m (\dot{x}_1^2 + 2 \dot{x}_1 \dot{x}_2 \cos\theta + \dot{x}_2^2)}$$

3) PE: $V_g = -mg\,x_2 \sin\theta$

↳ wrt the top of the ramp

4) $L = T - V$

$= T_M + T_m - V$

$= \frac{1}{2}M\dot{x}_1^2 + \frac{1}{2}m\left(\dot{x}_1^2 + 2\dot{x}_1\dot{x}_2\cos\theta + \dot{x}_2^2\right) - mg\,x_2\sin\theta$

5) $\gamma = \frac{d}{dt}\cdot\frac{dL}{d\dot{q}} - \frac{dL}{dq}$

$\frac{dL}{d\dot{x}_1} = M\dot{x}_1 + m\dot{x}_1 + m\dot{x}_2\cos\theta$

$\frac{d}{dt}\left(\frac{dL}{d\dot{x}_1}\right) = M\ddot{x}_1 + m\ddot{x}_1 + m\ddot{x}_2\cos\theta$

$\frac{dL}{dx_1} = 0$

$\frac{dL}{d\dot{x}_2} = m\dot{x}_1\cos\theta + m\dot{x}_2$

$\frac{d}{dt}\left(\frac{dL}{d\dot{x}_2}\right) = m\ddot{x}_1\cos\theta + m\ddot{x}_2$

$\frac{dL}{dx_2} = -mg\sin\theta$

$$\begin{bmatrix} F_{x_1} \\ F_{x_2} \end{bmatrix} = \begin{bmatrix} M\ddot{x}_1 + m\ddot{x}_1 + m\ddot{x}_2 \cos\theta \\ m\ddot{x}_1 \cos\theta + m\ddot{x}_2 - mg\sin\theta \end{bmatrix}$$

Control

$$F_{x_1} = (M+m)\,\ddot{x}_1 + m\ddot{x}_2 \cos\theta$$

$$F_{x_2} = m\ddot{x}_1 \cos\theta + m\ddot{x}_2 - mg\sin\theta$$

$\downarrow$

Plug in $\ddot{x}_1^d$ and $\ddot{x}_2^d$

Get forces necessary to apply in each direction

Add feedback

$\downarrow$

$$\boxed{\begin{aligned} F_{x_1} &= (M+m)\,\ddot{x}_1^d + m\ddot{x}_2^d \cos\theta + K_p(x_1^d - x_1) + K_d(\dot{x}^d - \dot{x}_1) + \cdots \\ F_{x_2} &= m\ddot{x}_1 \cos\theta + m\ddot{x}_2 - mg\sin\theta + K_p(x_2^d - x_2) + \cdots \end{aligned}}$$