EE106A Discussion 9: Dynamics

1 Dynamics

We've already learned kinematics, the study of motion. Now we're learning dynamics, the study of how forces affect that motion. The Newtonian dynamics you learned in high school work well in inertial reference frames. However, we deal with rotating reference frames which are not inertial, and dynamics becomes harder. Let's look at the difference.

1.1 Newton's Laws

Newton's three laws of motion form the cornerstone of dynamics. They are:

1. Every object in a state of uniform motion will remain in that state of motion unless an external force acts on it.
2. Force equals mass times acceleration.
3. For every action there is an equal and opposite reaction.

Newton's first law is called the law of inertia. If a mass is moving at a constant velocity, it will continue moving at that velocity unless a force acts upon it.

1.2 Inertial Frames

An inertial reference frame is a frame which follows Newton's first law: if a force isn't applied to a body, it will not accelerate. Thus, inertial frames do not themselves accelerate. However, inertial frames can move at constant velocity.

Problem 1:

1. *Imagine that you drop a bowling ball off the Leaning Tower of Piza. Is a frame attached to the ball an inertial frame?*
2. *Imagine a rocket flying through deep space at 100,000 mph. Is a frame attached to the rocket an inertial frame?*
3. *Imagine a pebble sitting on the ground on Earth. Is a frame attached to the pebble an inertial frame?*

1. No, a ball in freefall is not an inertial frame, since it's accelerating due to the force of gravity. A point on the Leaning Tower of Piza would appear to accelerate upwards, even though the net force on the tower is zero.
2. Yes, a rocket shooting through deep space is indeed an inertial frame. While it's going very fast, it's not accelerating and therefore it's an inertial frame (it's still much slower than light, so physics still works normally. If it were a hundred times faster, weird stuff would happen)

3. Kind of. The Earth is spinning about its axis, and orbiting the sun, which itself is orbiting the center of our galaxy. However, if the objects and motions we are concerned with are very small compared to earth, we can ignore these accelerations and pretend that the frame is inertial. If you're working with planes or satellites or planets, you have no such luck.

1.3 Rotating Reference Frames

Let's look at a rotating reference frame and figure out why it's not inertial. Intuitively, one might think that a frame rotating at a constant angular velocity would count as a non-accelerating frame, but that's not actually true.

Imagine a spinning disc and take two distinct points along a radius. Observed from a non-rotating reference frame, the two points have different velocities, while in the rotating frame both have zero velocity. In addition, if observed from the nonrotating frame the points will change directions (a change in velocity) even though their speed remains the same. This is due to *centripetal force* which pulls the rotating point towards the axis of rotation and causes the spinning.

Problem 2: *Prove that the centripetal force is defined* $\boldsymbol{F}_c = -mr\dot{\theta}^2\boldsymbol{r}$

There are two ways to calculate this. First with calculus (which only works in 2D) and second with the vector math we learned in the first half of this course (which requires a somewhat obscure cross product formula).

For the calculus method we express $\boldsymbol{r}$ as a cartesian vector

$$\boldsymbol{r} = \begin{bmatrix} r\cos(\theta) \\ r\sin(\theta) \end{bmatrix}$$

. Here r is a scalar, as is $\dot{\theta}$. We then define velocity

$$\boldsymbol{v} = \frac{d}{dt}\boldsymbol{r} = \begin{bmatrix} -r\sin(\theta)\dot{\theta} \\ r\cos(\theta)\dot{\theta} \end{bmatrix}$$

We define momentum $\boldsymbol{p} = m\boldsymbol{v}$ and take the derivative. Since r and $\dot{\theta}$ are constants we don't need to do a product rule expansion.

$$\boldsymbol{F}_c = \frac{d}{dt}\boldsymbol{p} = m\begin{bmatrix} -r\cos(\theta)\dot{\theta}^2 \\ -r\sin(\theta)\dot{\theta}^2 \end{bmatrix}$$

We can factor out some constants to get

$$\boldsymbol{F}_c = -m\dot{\theta}^2\begin{bmatrix} r\cos(\theta) \\ r\sin(\theta) \end{bmatrix} = -m\dot{\theta}^2\boldsymbol{r}$$

For the vector method we first define linear momentum

$$\boldsymbol{p} = m\boldsymbol{v} = m\boldsymbol{\omega} \times \boldsymbol{r}(t)$$

where $r(t)$ is the radial distance from the axis of rotation to the point in question and differentiate to get

$$\boldsymbol{F}_c = \frac{d\boldsymbol{p}}{dt} = m\frac{d\boldsymbol{v}}{dt} = m\boldsymbol{\omega} \times (\frac{d\boldsymbol{r}}{dt}) = m\boldsymbol{\omega} \times (\boldsymbol{v}) = m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{r})$$

Finally we use the vector triple product $(\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c}) = (\boldsymbol{a} \cdot \boldsymbol{c})\boldsymbol{b} - (\boldsymbol{a} \cdot \boldsymbol{b})\boldsymbol{c})$ to get

$$\boldsymbol{F}_c = m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{r}) = m\boldsymbol{\omega}^T\boldsymbol{r}\boldsymbol{\omega} - m\boldsymbol{\omega}^T\boldsymbol{\omega}\boldsymbol{r}$$

Since the radius is always perpendicular to the axis of rotation, $\boldsymbol{\omega}^T\boldsymbol{r} = 0$. Thus we're left with

$$\boldsymbol{F}_c = -m\boldsymbol{\omega}^T\boldsymbol{\omega}\boldsymbol{r} = -m|\boldsymbol{\omega}|_2^2\boldsymbol{r} = -m\dot{\theta}^2\boldsymbol{r}$$

Imagine a car driving quickly (at constant speed) along a circular freeway exit ramp. The car is accelerating towards the center of the circle – a centripetal acceleration – so there must be some force that causes this acceleration. In this case, it's the frictional force of the road pushing the car inwards. However, if you sit in the car, you're in a rotating reference frame. In this frame the car is not accelerating, but there's still a force pointing inwards from the tires. Since the net force on the car in this frame must be zero, we need to create a "fictitious force" pushing outwards. This is called the centrifugal force, and it's the "force" that pushes you to the side of your car when you take turns fast.

There's another fictitious force we need to worry about called the Coriolis force. Imagine that you're looking at Earth from space, and see an anchored zeppelin on the equator facing North. When you're standing on Earth, the zeppelin appears stationary, but from space, it looks like it's moving East at 1000 mph (since Earth rotates Eastward). Now imagine that the zeppelin releases its anchor and starts moving North. Since the Earth is a sphere, the effective radius decreases the farther North you go, and so the speed of the ground decreases proportionally. However, the zeppelin is still moving East at 1000 mph, while the ground speed decreases continuously as the zeppelin moves North. Thus, from the ground the zeppelin will seem to accelerate Eastward as it moves North.
The Coriolis force is generally quite small compared to other forces (an object dropped 50m at the equator will be deflected 7.7mm by the Coriolis force), so often we can ignore it.

1.4 Newton-Euler Dynamics in non-inertial frames

You can see these fictitious forces crop up in the body frame Newton-Euler dynamics equations:

$$\begin{bmatrix} mI & 0 \\ 0 & \mathcal{I} \end{bmatrix} \begin{bmatrix} \dot{v}^b \\ \dot{\omega}^b \end{bmatrix} + \begin{bmatrix} \omega^b \times mv^b \\ \omega^b \times \mathcal{I}\omega^b \end{bmatrix} = F^b$$

The top term in the right matrix is the Coriolis term. Centrifugal terms will only appear if you define a new frame with constant displacement from the body frame. This is one of the reasons that Newton-Euler equations become difficult to deal with in multibody dynamics problems.

2 Lagranginan Dynamics

In Lagrangian dynamics we'll be using conservation of energy to derive the equations of motions of our robot. Since energy is a scalar term, it's invariant to our system parameterization – we can use whichever coordinates we want. Thus, the first step in any Lagrangian dynamics problem is to choose a set of generalized coordinates q. In order to minimize system constraints, we want to choose the minimum number of coordinates to represent our system. For open-chain robot manipulators, we'll generally choose our joint angles, so $q = \theta$ and $\dot{q} = \dot{\theta}$.

2.1 System Energy

Once we have our generalized coordinates q, we must define the kinetic and potential energies of our system in terms of those coordinates and their derivatives.
The kinetic energy $T(q, \dot{q})$ is the sum of the kinetic energies of each rigid body T_i. It will depend on the derivatives of our generalized coordinates $\dot{q}$, and potentially on the coordinates q themselves. Kinetic energy is composed of both rotational and translational terms, so

$$T_i = \frac{1}{2}I\dot{\theta}^2 + \frac{1}{2}mv^2$$

This is the same as using the generalized inertia matrix of the rigid body expressed in the body frame, as well as the body velocity.

$$T_i = \frac{1}{2} {V_i^b}^T \mathcal{M}_i V_i^b$$

where the generalized inertia matrix in the body frame

$$\mathcal{M}_i = \begin{bmatrix} m_i I_3 & 0 \\ 0 & \mathcal{I}_i \end{bmatrix}$$

All we're doing here is combining the mass (by which we multiply a 3×3 identity matrix I_3) and the moment of inertia of the system into a single matrix. Note that however you define your kinetic energy, you'll need to represent it as a function of q and $\dot{q}$. You'll see how to do this below.

The potential energy $V(q)$ is also the sum of of the potential energies of each rigid body V_i. It will only depend on our state, not its derivatives. There are two sources of potential energy we normally deal with: gravity and springs. The potential energy due to gravity is

$$V_g = mgh$$

where h is the height of the center of mass of the rigid body. The potential energy due to springs is

$$V_s = \frac{1}{2} k x^2$$

where x is the displacement of the spring from its neutral position. Once again, note that you'll need to represent height h and displacement x as functions of q.

2.2 The Lagrangian and Equations of Motion

We can now define the Lagrangian

$$L = T - V = \sum T_i - \sum V_i$$

which is the difference in kinetic and potential energy of the system. Once we can do this, we can generate the equations of motion of the system through

$$\Upsilon = \frac{d}{dt} \frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q}$$

Here Υ is the vector of generalized forces, the force or torque terms that correspond to each generalized coordinate.

2.3 Lagrangian Dynamics of an Open Chained Manipulator

For an open chained manipulator we choose the joint angles θ as our generalized coordinates. We then start by defining our kinetic energy T. We model the robot as a set of rigid bodies, or links, connected together by joints. We know that the kinetic energy of a rigid body is

$$T_i = \frac{1}{2} {V_i^b}^T \mathcal{M}_i V_i^b$$

However, we need to represent these in terms of θ and $\dot{\theta}$. To do so we use the body Jacobian.

$$V^b = J^b(\theta)\dot{\theta}$$

Our equation for kinetic energy then becomes

$$T_i(\theta,\dot{\theta}) = \frac{1}{2}\dot{\theta}^T {J_i^b}^T(\theta)\mathcal{M}_i J_i^b(\theta)\dot{\theta}$$

We know that

$$T(\theta,\dot{\theta}) = \sum T_i(\theta,\dot{\theta}) = \sum \frac{1}{2}\dot{\theta}^T {J_i^b}^T(\theta)\mathcal{M}_i J_i^b(\theta)\dot{\theta}$$

We then consolidate this to get

$$T(\theta,\dot{\theta}) = \frac{1}{2}\dot{(\theta)}^T M(\theta)\dot{\theta}$$

where the manipulator inertia matrix $M(\theta)$ is defined

$$\sum {J_i^b}^T(\theta)\mathcal{M}_i J_i^b(\theta)$$

What we're doing here is we're expressing the mass and inertias of each rigid body as moments of inertia about each joint axis (or masses along it if we have prismatic joints).
The potential energy is expressed a bit more simply as

$$V(\theta) = \sum V_i(\theta) = \sum m_i g h_i(\theta)$$

where $h_i(\theta)$ is the height of the center of mass of the ith link as a function of θ.
The Lagrangian is thus

$$L(\theta,\dot{\theta}) = \frac{1}{2}\dot{(\theta)}^T M(\theta)\dot{\theta} - V(\theta)$$

2.4 Equations of Motion of an Open Chain Manipulator

We know that the equations of motion are

$$\Upsilon = \frac{d}{dt}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q}$$

When we take those derivatives we get four terms:

$$\Upsilon = M(\theta)\ddot{\theta} + \dot{M}(\theta)\dot{\theta} - \frac{1}{2}\dot{\theta}^T\frac{\partial M(\theta)}{\partial\theta}\dot{\theta} + \frac{\partial V(\theta)}{\partial\theta}$$

The first two are a result of the product rule on the time derivative of $\frac{\partial L}{\partial\dot{\theta}} = M(\theta)\dot{\theta}$. The second two come from $\frac{\partial L}{\partial\theta} = \frac{\partial T(\theta,\dot{\theta})}{\partial\theta} - \frac{\partial V(\theta)}{\partial\theta}$.

Without focusing too hard on the actual computation of the terms (generally you'll have a computer do it for you) we can combine the middle two terms $\dot{M}(\theta)\dot{\theta} - \frac{1}{2}\dot{\theta}^T\frac{\partial M(\theta)}{\partial\theta}\dot{\theta} = C(\theta,\dot{\theta})\dot{\theta}$ and name the fourth term $\frac{\partial V(\theta)}{\partial\theta} = G(\theta)$. We now have the dynamics in their final form:

$$\Upsilon = \tau = M(\theta)\ddot{\theta} + C(\theta,\dot{\theta})\dot{\theta} + G(\theta)$$

In this case, our generalized force vector Υ is the vector of joint torques τ. The first term on the right hand side is the manipulator inertia matrix (which we've already seen) times the angular acceleration of the joints. The second term we call the Coriolis matrix, which is multiplied by the angular velocity of the joints. This term collects all the fictitious forces that arise from spinning reference frames and collects them into one term. And the third term we call the gravity vector.

3 General Equations of Motion and Examples

In fact, the Lagrangian dynamics of any system can be represented in this form:

$$\Upsilon = M(q)\ddot{q} + C(q,\dot{q})\dot{q} + G(q)$$

The inertia matrix represents the inertia (mass and moment of inertia) of the system in the coordinate system of our generalized coordinates. The Coriolis matrix contains all the fictitious forces arising from rotating reference frames. And the gravity vector is the effect of the potential energy forces (gravity and springs) on each of our generalized coordinates.

3.1 The Inertia Matrix: Spinning Teacups

We have a spinning teacup ride with two teacups (disks) mounted to the spinning ride (another disk). The teacups each spin about their own axes. The ride itself has a mass and moment of inertia. Each teacup is mounted a distance L from the center of the ride, and each has its own mass and moment of inertia.

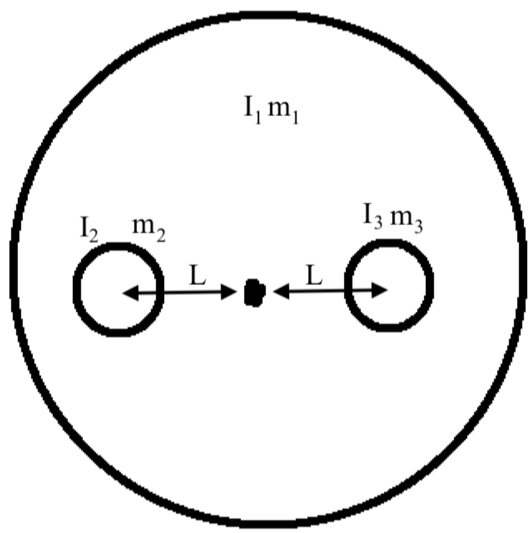

Figure 1: Idealized Teacup Ride

Problem 3: *What are the equations of motion of this system? What are the inertia matrix, coriolis matrix and gravity vector? What are the generalized forces?*
First we define our generalized coordinates. Since the ride itself can't move, we only need the angle of each cup (relative to some arbitrary zero position) to define the position of our system. Thus $q = [\theta_1, \theta_2, \theta_3]^T$.

Now we define the kinetic energy and potential energies of the system:

$$T = \frac{1}{2}I_1\dot{\theta}_1^2 + \frac{1}{2}m_1 v_1^2 + \frac{1}{2}I_2\dot{\theta}_2^2 + \frac{1}{2}m_2 v_2^2 + \frac{1}{2}I_3\dot{\theta}_3^2 + \frac{1}{2}m_3 v_3^2$$

Since the system doesn't move in place, $v_1 = 0$. We can use polar coordinates to calculate the velocities of m_2 and m_3:

$$v_2 = v_3 = L\dot{\theta}_1$$

Thus the kinetic energy is

$$T = \frac{1}{2}(I_1 + m_2L^2 + m_3L^2)\dot{\theta}_1^2 + \frac{1}{2}I_2\dot{\theta}_2^2 + \frac{1}{2}I_3\dot{\theta}_3^2$$

Since we have no gravity or springs in this system, our potential energy is zero:

$$V = 0$$

So our Lagrangian is

$$L = T - V = T = \frac{1}{2}(I_1 + m_2L^2 + m_3L^2)\dot{\theta}_1^2 + \frac{1}{2}I_2\dot{\theta}_2^2 + \frac{1}{2}I_3\dot{\theta}_3^2$$

Now we take derivatives:

$$\frac{\partial L}{\partial \dot{q}} = \begin{bmatrix} (I_1 + m_2L^2 + m_3L^2)\dot{\theta}_1 \\ I_2\dot{\theta}_2 \\ I_3\dot{\theta}_3 \end{bmatrix}$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} = \begin{bmatrix} (I_1 + m_2L^2 + m_3L^2)\ddot{\theta}_1 \\ I_2\ddot{\theta}_2 \\ I_3\ddot{\theta}_3 \end{bmatrix}$$

$$\frac{\partial L}{\partial q} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Thus our equations of motion are:

$$\Upsilon = \begin{bmatrix} (I_1 + m_2L^2 + m_3L^2)\ddot{\theta}_1 \\ I_2\ddot{\theta}_2 \\ I_3\ddot{\theta}_3 \end{bmatrix}$$

which is

$$\Upsilon = \begin{bmatrix} (I_1 + m_2L^2 + m_3L^2) & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{bmatrix} \begin{bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \\ \ddot{\theta}_3 \end{bmatrix}$$

Our inertia matrix is:

$$M(q) = \begin{bmatrix} (I_1 + m_2L^2 + m_3L^2) & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{bmatrix}$$

Our Coriolis matrix is zero, as is our gravity vector. Our generalized forces are the torques about each rotating axis:

$$\Upsilon = \begin{bmatrix} \tau_1 \\ \tau_2 \\ \tau_3 \end{bmatrix}$$

You might be asking: "there are spinning bodies connected to other spinning bodies here. Why don't we have centripedal or Coriolis forces?". The reason is that the inertia matrix (which we could have defined back when we calculated the kinetic energy) is invariant to changes in our generalized coordinates; M doesn't depend on q. Thus, inertial effects can't actually cause motion in our system. While inertial forces certainly still exist, they're resisted by the constraint forces in our system (the mechanical structure holding the teacup ride together) and thus don't affect our system dynamics at all.

3.2 The Coriolis Matrix: A sliding mass on a carousel

We have a spinning carousel (disk) with inertia I. On this carousel is a massless, frictionless linear rail that lies on a radius of the carousel. On this rail is a point mass with mass m.

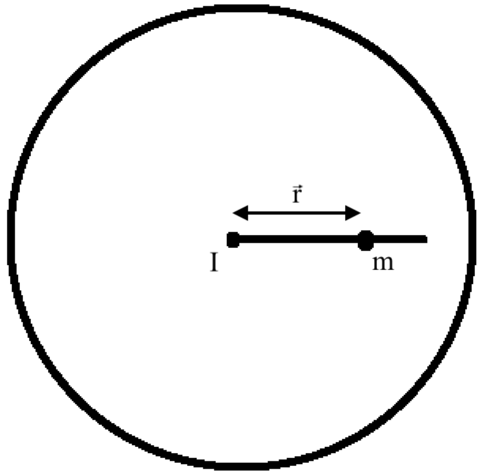

Figure 2: Carousel with beam and point mass

Problem 4: *What are the equations of motion of this system? What are the inertia matrix, coriolis matrix and gravity vector? What are the generalized forces?*
First we define our generalized coordinates. We can parameterize the entire system pose using the angle of the rail θ with respect to some arbitrary zero and the position of the mass along the rail r. Thus $q = [\theta, r]^T$.

Now we define the kinetic energy and potential energies of the system:

$$T = \frac{1}{2}I\dot{\theta} + \frac{1}{2}mv^2$$

The carousel doesn't move in place so we can ignore its translational kinetic energy. The point mass has no inertia so we can ignore its rotational kinetic energy. To calculate the translational kinetic energy of the point mass v in terms of our generalized coordinates r and θ we'll use a cartesian representation (you can use any representation and it'll come out the same)

$$x = r\cos(\theta)$$
$$y = r\sin(\theta)$$
$$\dot{x} = \dot{r}\cos(\theta) - r\sin(\theta)\dot{\theta}$$
$$\dot{y} = \dot{r}\sin(\theta) + r\cos(\theta)\dot{\theta}$$
$$v = \dot{x}^2 + \dot{y}^2 = \dot{r}^2\cos^2(\theta) - 2\dot{r}\cos(\theta)r\sin(\theta)\dot{\theta} + r^2\sin^2(\theta)\dot{\theta}^2 + \dot{r}^2\sin^2(\theta) + 2\dot{r}\sin(\theta)r\cos(\theta)\dot{\theta} + r^2\cos^2(\theta)\dot{\theta}^2$$
$$v = \dot{r}^2 + r^2\dot{\theta}^2$$

Thus the kinetic energy is

$$T = \frac{1}{2}I\dot{\theta}^2 + \frac{1}{2}mr^2\dot{\theta}^2 + \frac{1}{2}m\dot{r}^2$$

Since we have no gravity or springs in this system, our potential energy is zero:

$$V = 0$$

So our Lagrangian is

$$L = T - V = T = \frac{1}{2}I\dot{\theta}^2 + \frac{1}{2}mr^2\dot{\theta}^2 + \frac{1}{2}m\dot{r}^2$$

Now we take derivatives:

$$\frac{\partial L}{\partial \dot{q}} = \begin{bmatrix} \frac{\partial L}{\partial \dot{\theta}} \\ \frac{\partial L}{\partial \dot{r}} \end{bmatrix} = \begin{bmatrix} I\dot{\theta} + mr^2\dot{\theta} \\ m\dot{r} \end{bmatrix}$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} = \begin{bmatrix} I\ddot{\theta} + 2mr\dot{r}\dot{\theta} + mr^2\ddot{\theta} \\ m\ddot{r} \end{bmatrix}$$

$$\frac{\partial L}{\partial q} = \begin{bmatrix} \frac{\partial L}{\partial \theta} \\ \frac{\partial L}{\partial r} \end{bmatrix} = \begin{bmatrix} 0 \\ mr\dot{\theta}^2 \end{bmatrix}$$

Thus our equations of motion are:

$$\Upsilon = \begin{bmatrix} I\ddot{\theta} + 2mr\dot{r}\dot{\theta} + mr^2\ddot{\theta} \\ m\ddot{r} - mr\dot{\theta}^2 \end{bmatrix}$$

which is

$$\Upsilon = \begin{bmatrix} I + mr^2 & 0 \\ 0 & m \end{bmatrix} \begin{bmatrix} \ddot{\theta} \\ \ddot{r} \end{bmatrix} + \begin{bmatrix} 2mr\dot{r} & 0 \\ -mr\dot{\theta} & 0 \end{bmatrix} \begin{bmatrix} \dot{\theta} \\ \dot{r} \end{bmatrix}$$

Our inertia matrix is:

$$M(q) = \begin{bmatrix} I + mr^2 & 0 \\ 0 & m \end{bmatrix}$$

Our Coriolis matrix is:

$$C(q, \dot{q}) = \begin{bmatrix} 2mr\dot{r} & 0 \\ -mr\dot{\theta} & 0 \end{bmatrix}$$

Since we've calculated things a bit differently from the way the book does it, we don't get the centripetal/centrifugal terms on the diagonal and the Coriolis terms off-diagonal. However, we can identify $-mr\dot{\theta}^2$ as a centrifugal force since we have $\dot{q}_i^2$, and $2mr\dot{r}\dot{\theta}$ as a Coriolis force since we have $\dot{q}_i\dot{q}_j$.

Our gravity vector is zero here since we have no potential energy component. Our generalized forces are the torques about the center of the carousel as well as the force on the point mass along the rail.

$$\Upsilon = \begin{bmatrix} \tau \\ f \end{bmatrix}$$

3.3 The Gravity Vector: A mass spring

We have an object with mass m hanging from a spring with stiffness k. The object is on a frictionless rail and is constrained to only move in the vertical direction.

Problem 5: *What are the equations of motion of this system? What are the inertia matrix, coriolis matrix and gravity vector? What are the generalized forces?*

First we define our generalized coordinates. This is a one dimensional system, and can be completely parameterized by the distance of the object on the spring from the spring's "zero position". We'll call this distance x. Thus $q = x$.

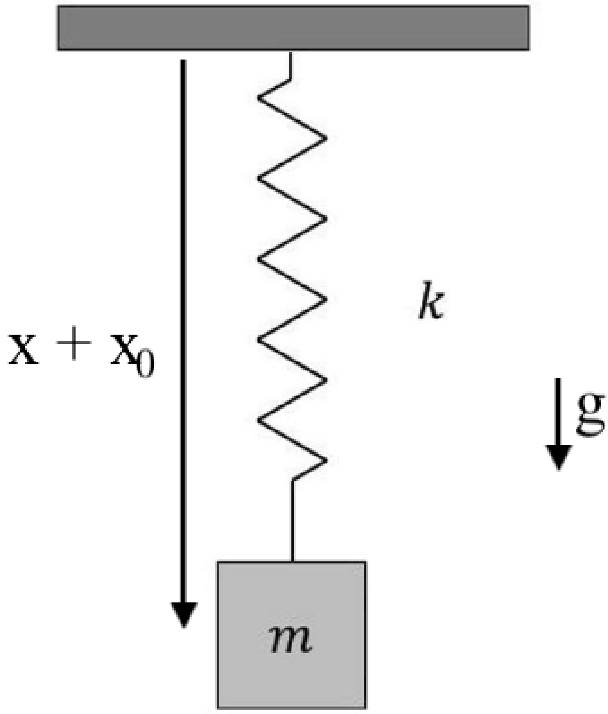

Figure 3: Mass Spring

Now we define the kinetic energy and potential energies of the system:

$$T = \frac{1}{2}mv^2 = \frac{1}{2}m\dot{x}^2$$

The potential energy of our system is the potential energy due to gravity plus the potential energy of the spring. For ease of calculation, we define the height to be zero when the object sits at the spring's unextended position (we can choose the zero anywhere we want without it affecting the dynamics).

$$V = -mgx + \frac{1}{2}kx^2$$

So our Lagrangian is

$$L = T - V = \frac{1}{2}m\dot{x}^2 + mgx - \frac{1}{2}kx^2$$

Now we take derivatives:

$$\frac{\partial L}{\partial \dot{q}} = \frac{\partial L}{\partial \dot{x}} = m\dot{x}$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} = m\ddot{x}$$

$$\frac{\partial L}{\partial q} = \frac{\partial L}{\partial x} = mg - kx$$

Thus our equations of motion are:

$$\Upsilon = m\ddot{x} - mg + kx$$

Our inertia "matrix" is:

$$M(q) = m$$

Our Coriolis matrix is zero. Our gravity "vector" is

$$G(q) = -mg + kx$$

Our generalized force is the vertical force on the mass. Since we've defined x to increase as the mass moves downward, the positive direction of the force is also downwards

$$\Upsilon = F$$

Note that if we were to set the external force to zero, and rearrange some terms we'd get:

$$m\ddot{x} = mg - kx$$

Which is exactly what we'd read off a free body diagram if we were to draw one for this system.

3.4 Dynamics of a Cart, Pole, Spring System

Figure 4 shows a model of a cart balancing a rod of uniform mass m, length $2L$, and moment of inertia I at the center of mass about an axis pointing out of the plane of the paper. The cart has a mass M, is tethered by a spring of spring constant k, and has a motor that can exert an external force F on the center of mass of the cart. Let x denote the deviation of the cart from the uncompressed location of the spring. Mounted on the cart is a pendulum of length l of negligible mass with a ball of mass m at its end. The pendulum swivels about the center of mass of the cart with an angular deflection θ from the vertical. The cart contains a second motor that can exert an external torque τ on the pendulum.

Derive the Lagrangian equations of motion. In particular state the the generalized Inertia matrix, the Coriolis matrix, and the gravity vector for this system.

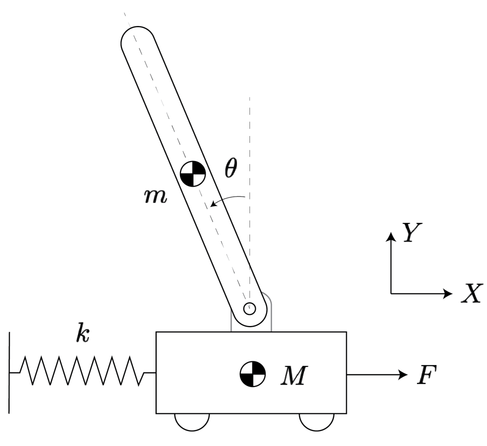

Figure 4: Cart Spring Pendulum system

The generalized coordinates we'll be using are $q = [x, \theta]^T$, and the corresponding generalized forces are $\Upsilon = [F, \tau]^T$. First, we find the kinetic energies of each mass. The cart doesn't move in the y direction or spin, so its kinetic energy is

$$T_M = \frac{1}{2}M\dot{x}^2$$

The center of mass of the rod moves in both x and y, so we'll use Cartesian coordinates:

$$p_x = x - L\sin(\theta)$$
$$p_y = L\cos(\theta)$$
$$v_x = \dot{x} - L\cos(\theta)\dot{\theta}$$
$$v_y = -L\sin(\theta)\dot{\theta}$$
$$v_m^2 = v_x^2 + v_y^2 = \dot{x}^2 - 2L\cos(\theta)\dot{\theta}\dot{x} + L^2\cos^2(\theta)\dot{\theta}^2 + L^2\sin^2(\theta)\dot{\theta}^2$$
$$v_m^2 = \dot{x}^2 - 2L\cos(\theta)\dot{\theta}\dot{x} + L^2\dot{\theta}^2$$

The kinetic energy of the rod has a translational component and a rotational component. It is

$$T_m = \frac{1}{2}mv_m^2 + \frac{1}{2}I\dot{\theta}^2 = \frac{1}{2}m\dot{x}^2 - mL\cos(\theta)\dot{\theta}\dot{x} + \frac{1}{2}mL^2\dot{\theta}^2 + \frac{1}{2}I\dot{\theta}^2$$

Now the total kinetic energy T of the system is

$$T = \frac{1}{2}(M+m)\dot{x}^2 + \frac{1}{2}(mL^2+I)\dot{\theta}^2 - mL\cos(\theta)\dot{x}\dot{\theta}$$

The two sources of potential energy are gravity and spring force. Earlier we defined the height y to be zero at the pivot point of the pendulum. We need to remain consistent so we do so again (this is the easiest place to define it). x is the distance from the uncompressed spring length. Thus we have

$$V = \frac{1}{2}kx^2 + mgL\cos(\theta)$$

We then define the Lagrangian:

$$L = T - V = \frac{1}{2}(M+m)\dot{x}^2 + \frac{1}{2}(mL^2+I)\dot{\theta}^2 - mL\cos(\theta)\dot{x}\dot{\theta} - \frac{1}{2}kx^2 - mgL\cos(\theta)$$

Now we just do a bunch of derivatives:

$$\frac{\partial L}{\partial \dot{q}} = \begin{bmatrix} \frac{\partial L}{\partial \dot{x}} \\ \frac{\partial L}{\partial \dot{\theta}} \end{bmatrix} = \begin{bmatrix} (M+m)\dot{x} - mL\cos(\theta)\dot{\theta} \\ -mL\cos(\theta)\dot{x} + (mL^2+I)\dot{\theta} \end{bmatrix}$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} = \begin{bmatrix} (M+m)\ddot{x} - mL\cos(\theta)\ddot{\theta} + mL\sin(\theta)\dot{\theta}^2 \\ mL\sin(\theta)\dot{\theta}\dot{x} - mL\cos(\theta)\ddot{x} + (mL^2+I)\ddot{\theta} \end{bmatrix}$$

And finally

$$\frac{\partial L}{\partial q} = \begin{bmatrix} \frac{\partial L}{\partial x} \\ \frac{\partial L}{\partial \theta} \end{bmatrix} = \begin{bmatrix} -kx \\ mL\sin(\theta)\dot{\theta}\dot{x} + mgL\sin(\theta) \end{bmatrix}$$

We put them together to get

$$\begin{aligned}
\Upsilon = \begin{bmatrix} F \\ \tau \end{bmatrix} &= \frac{d}{dt}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} \\
&= \begin{bmatrix} (M+m)\ddot{x} - mL\cos(\theta)\ddot{\theta} + mL\sin(\theta)\dot{\theta}^2 \\ mL\sin(\theta)\dot{\theta}\dot{x} - mL\cos(\theta)\ddot{x} + (mL^2+I)\ddot{\theta} \end{bmatrix} - \begin{bmatrix} -kx \\ mL\sin(\theta)\dot{\theta}\dot{x} + mgL\sin(\theta) \end{bmatrix} \\
&= \begin{bmatrix} (M+m)\ddot{x} - mL\cos(\theta)\ddot{\theta} + mL\sin(\theta)\dot{\theta}^2 + kx \\ -mL\cos(\theta)\ddot{x} + (mL^2+I)\ddot{\theta} - mgL\sin(\theta) \end{bmatrix}
\end{aligned}$$

We break this up into M, C, and G matrices.

$$M(q) = \begin{bmatrix} m+M & -mL\cos(\theta) \\ -mL\cos(\theta) & mL^2+I \end{bmatrix}$$

$$C(q,\dot{q}) = \begin{bmatrix} 0 & mL\sin(\theta)\dot{\theta} \\ 0 & 0 \end{bmatrix}$$

and finally

$$G(q) = \begin{bmatrix} kx \\ -mgL\sin(\theta) \end{bmatrix}$$