EE106A Discussion 8: Jacobians

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1 Overview

- velocity: relative to stationary frame
- Express in terms of spatial or body frame

Last week, we learned about spatial and body velocity twists between two frames A and B. These velocity twists are useful because they allow us to find the instantaneous velocity of the B frame expressed in both spatial and body coordinates.

$$v_{q_a}(t) := \dot{q}_a(t) = \dot{g}_{ab}(t)q_b = \underbrace{\dot{g}_{ab}(t)g_{ab}^{-1}(t)}_{:=\widehat{V}_{ab}^s} q_a = \widehat{V}_{ab}^s q_a \quad \rightarrow \text{Spatial} \tag{1}$$

Just a coordinate switch

$$v_{q_b}(t) := g_{ab}^{-1}(t)v_{q_a}(t) = \underbrace{g_{ab}^{-1}(t)\dot{g}_{ab}}_{:=\widehat{V}_{ab}^b} q_b = \widehat{V}_{ab}^b q_b \quad \rightarrow \text{Body} \tag{2}$$

Today, we will be thinking of velocities in the context of robotic manipulators. We will be finding the velocities between the fixed frame S and the end effector frame T, $\widehat{V}_{st}^s$ and $\widehat{V}_{st}^b$.

To do so, we will introduce the notion of spatial and body manipulator Jacobians. Then, we will see how these manipulator Jacobians help us detect singular configurations.

2 Adjoint for Twist Coordinate Change

When working with twists, we can transform a twist matrix $\widehat{\xi}$ into a different coordinate system defined by g, so that it becomes $\widehat{\xi}'$

$$\widehat{\xi}' = g\widehat{\xi}g^{-1} \tag{3}$$

In twist coordinates,

$$\xi' = Ad_g\xi \tag{4}$$

$$Ad_g = \begin{bmatrix} R & \hat{p}R \\ 0 & R \end{bmatrix} \in \mathbb{R}^{6\times 6}$$

$$\boxed{V_{ab}^s = Ad_{g_{ab}} \cdot V_{ab}^b}$$

$$\boxed{V_{ab}^b = Ad_{g_{ab}}^{-1} V_{ab}^s}$$

$$Ad_g^{-1} = Ad_{g^{-1}}$$

Jacobians: - Tackle problem of having multiple links
- We know the speed of each link ($\dot{\theta}$) (sensors inside our robot)
- What is the end effector velocity? ↳ as a function of each joint

3 Spatial Jacobians

3.1 Definition

As before, we have the expression for $\widehat{V}_{st}^s$ as a function of the transformation between S and T:

$$\widehat{V}_{st}^s = \dot{g}_{st}(\theta)g_{st}^{-1}(\theta) \tag{5}$$

In twist coordinates,

$$\begin{bmatrix} v^s \\ \omega^s \end{bmatrix} \leftarrow V_{st}^s = J_{st}^s(\theta)\dot{\theta} \tag{6}$$

→ End effector velocity broken up by joint

- Tells you end effector velocity as a function of time broken down by joint

where the *spatial manipulator Jacobian* $J_{st}^s(\theta)$ is defined as

- Multiply contribution of each joint by speed of joint

$$J_{st}^s(\theta) = \left[\left(\frac{\partial g_{st}}{\partial \theta_1}\right)^\vee \quad \ldots \quad \left(\frac{\partial g_{st}}{\partial \theta_n}\right)^\vee \right] \tag{7}$$

→ Break down $\dot{g}_{st}$ into each θ

$$= \left[\xi_1 \quad \xi_2' \quad \ldots \quad \xi_n' \right] \tag{8}$$

$\xi_1' = \xi_1$

$$\xi_i' = \underbrace{Ad_{(e^{\hat{\xi}_1\theta_1}\ldots e^{\hat{\xi}_{i-1}\theta_{i-1}})}}_{\text{Changes the configuration of each joint}}\xi_i \tag{9}$$

3.2 Interpretation

For some configuration θ, the spatial manipulator Jacobian maps the joint velocity vector $\dot{\theta}$ into the spatial velocity twist coordinates of the end-effector.

The i^{th} column of the spatial Jacobian ξ_i' is equal to the i^{th} joint twist transformed to the current manipulator configuration and written in spatial coordinates.

Problem 1. *Explain how this physical interpretation is true.*

Jacobian:
1. Find each ξ_i'
2. Put them in a matrix
3. Find end effector velocity

- ξ_i → i^{th} joint twist in the reference configuration
- Transformed by $\xi_1 \ldots \xi_{i-1}$
  ↳ orientation of ξ_i changes
  $e^{\hat{\xi}_1\theta_1} \ldots e^{\hat{\xi}_{i-1}\theta_{i-1}}$ → The change ξ_i undergoes
- Adjoints transform twists → get new orientation
  $Ad_{(e^{\hat{\xi}_1\theta_1} \ldots e^{\hat{\xi}_{i-1}\theta_{i-1}})} \xi_i = \xi_i'$

3.3 How it's used

We can use the spatial Jacobian to compute the instantaneous velocity of a point q attached to the end-effector relative to the spatial frame. This velocity is

$$v_{q_s} = \widehat{V}_{st}^s q_s = (J_{st}^s(\theta)\dot{\theta})^\wedge q_s \tag{10}$$

where q_s is the coordinates of q in the spatial frame.

- The Jacobian gives you the spatial velocity
- Breaks it down by current speeds of each joint

We know:

$$\hat{V}^s_{st} = \dot{g}_{st}(\theta)\, g^{-1}_{st}(\theta) \rightarrow$$ end-effector velocity (Body frame)

We want:

Velocity broken down by joint

Idea:

Use partial derivatives:

$$\dot{g}_{st} \qquad \underbrace{\left(\frac{\partial g_{st}}{\partial \theta_i}\right)}_{\text{Partial derivative of } g_{st} \text{ w.r.t. } \theta_i} g^{-1}_{st} \rightarrow$$ contribution of each θ_i to end-effector velocity

$$g_{st}(\theta) = e^{\hat{\xi}_1\theta_1} (\cdot\cdot) \; e^{\hat{\xi}_n\theta_n} \cdot g_{st}(0) \rightarrow \text{FK map}$$

$$\frac{\partial g_{st}}{\partial \theta_i} = e^{\hat{\xi}_1\theta_1} \cdots \left(\hat{\xi}_i \, e^{\hat{\xi}_i\theta_i}\right) \cdots e^{\hat{\xi}_n\theta_n} \; g_{st}(0)$$

$$(AB)^{-1} = B^{-1}A^{-1}$$

$$g^{-1}_{st} = g^{-1}_{st}(0)\, e^{-\hat{\xi}_n\theta_n} \cdots e^{-\hat{\xi}_1\theta_1}$$

$$\left(\frac{\partial g_{st}}{\partial \theta_i} \cdot g^{-1}_{st}\right) = \underbrace{e^{\hat{\xi}_1\theta_1} \cdots}\left(\hat{\xi}_i\, \cancel{e^{\hat{\xi}_i\theta_i}}\right)\cancel{\cdots e^{\hat{\xi}_n\theta_n}}\, \cancel{g_{st}(0)} \cdot \cancel{g^{-1}_{st}(0)}\, \cancel{e^{-\hat{\xi}_n\theta_n}} \cancel{\cdots}$$

$$= \underbrace{e^{\hat{\xi}_1 \theta_1} \cdots}_{g = e^{\hat{\xi}_1\theta_1} \cdots e^{\hat{\xi}_{i-1}\theta_{i-1}}} \hat{\xi}_i \underbrace{e^{-\hat{\xi}_{i-1}\theta_{i-1}} \cdots e^{-\hat{\xi}_1\theta_1}}_{g^{-1}}$$

$$\hat{\xi}' = g \hat{\xi} g^{-1}$$

$$\boxed{\left(\frac{\partial g_{st}}{\partial \theta_i} g_{st}^{-1} \right)^{\vee} = \underbrace{Ad_{\left(e^{\hat{\xi}_1\theta_1} \cdots e^{\hat{\xi}_{i-1}\theta_{i-1}} \right)} \xi_i}_{= \xi_i'}}$$

- Velocity contribution of ξ_i depends only on preceding ξs

$$J^s_{st}(\theta) = \begin{bmatrix} \xi_1' & \xi_2' & \cdots & \xi_n' \end{bmatrix}$$

$$\in \mathbb{R}^{6 \times n}$$

- Body velocity in terms of each joint!

$\xi \rightarrow$ twist in reference config Spatial frame

$\xi' \rightarrow$ twist after transform spatial frame

$\xi^{+} \rightarrow$ twist after transform body frame

4 Body Jacobians

4.1 Definition

Now let's look at velocity twists in the body frame rather than in the spatial frame:

$$\widehat{V}_{st}^{b} = g_{st}^{-1}(\theta)\dot{g}_{st}(\theta) \tag{11}$$

In twist coordinates,

$$V_{st}^{b} = J_{st}^{b}(\theta)\dot{\theta} \tag{12}$$

where the *body manipulator Jacobian* $J_{st}^{b}(\theta)$ is defined as

* Proof similar to above

$$J_{st}^{b}(\theta) = [\xi_1^{\dagger} \quad \xi_2^{\dagger} \quad \dots \quad \xi_n^{\dagger}] \tag{13}$$

$$\xi_i^{\dagger} = Ad^{-1}_{(e^{\widehat{\xi}_{i+1}\theta_{i+1}} \dots e^{\widehat{\xi}_n\theta_n} g_{st}(0))} \xi_i \tag{14}$$

- Inverse adjoint
- Exponentials of joints that come after

4.2 Interpretation

For some configuration θ, the body manipulator Jacobian maps the joint velocity vector $\dot{\theta}$ into the body velocity twist coordinates of the end-effector.

The i^{th} column of the body Jacobian $\xi_i^{\dagger}$ is equal to the i^{th} joint twist transformed to the current manipulator configuration and written in body coordinates.

Problem 2. *Explain how this physical interpretation is true.*

- ξ^{+} is ξ' (transformed ξ) in the body frame coords

$\xi^{+} = Ad_{g_{st}(\theta)} \cdot \xi'$

4.3 How it's used

We can use the body Jacobian to compute the instantaneous velocity of a point q attached to the end-effector relative to the body frame. This velocity is

$$v_{q_b} = \widehat{V}_{st}^{b} q_b = (J_{st}^{b}(\theta)\dot{\theta})^{\wedge} q_b \tag{15}$$

where q_b is the coordinates of q in the tool frame.

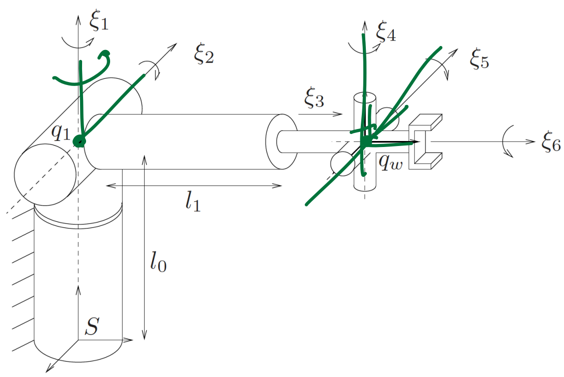
same idea - still a velocity Broken down in terms of joints In terms of body frame

4.4 Converting between Spatial and Body Jacobians

$$J_{st}^{s}(\theta) = Ad_{g_{st}(\theta)} J_{st}^{b}(\theta) \tag{16}$$

$Ad^{-1}_{g_{st}(\theta)} J^{s}_{st}(\theta) = J^{b}_{st}(\theta)$

Problem 3. *Find the spatial and body manipulator Jacobians for the Stanford manipulator.*



$$e_1 = e^{\hat{\xi}_1 \theta_1}$$

Figure 1: Stanford manipulator

$$J^s_{st}(\theta) = \begin{bmatrix} \xi_1' & \cdots & \xi_6' \end{bmatrix}$$

$$= \begin{bmatrix} \xi_1 & Ad_{e_1} \xi_2 & \cdots & Ad_{e_1 \cdots e_5} \xi_6 \end{bmatrix}$$

$\downarrow$

Adjoints change direction of joints based on preceding joints

$$J^s_{st}(\theta) = \begin{bmatrix} -\omega_1 \times q_1 & -\omega_2' \times q_1' & \cdots & -\omega_6' \times q_w' \\ \omega_1 & \omega_2' & \cdots & \omega_6' \end{bmatrix}$$

New direction of $\omega_2 \to$ affected by ξ_1

$$\omega_2' = e^{\hat{\omega}_1 \theta_1} \cdot \omega_2$$

$$= e^{\hat{\omega}_1 \theta_1} \cdot \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}$$

$$q_1' = q_1$$

ξ_1, ξ_2 won't change location of q_1

$$\xi_2' = \begin{bmatrix} -\omega_2' \times q_1' \\ \omega_2' \end{bmatrix}$$

$\xi_3 : \begin{bmatrix} v_3 \\ 0 \end{bmatrix}$ $\quad v_3' = e^{\hat{\omega}_1 \theta_1} e^{\hat{\omega}_2 \theta_2} \cdot v_3$

↓

New direction of prismatic joint

2 rotation matrices multiplied to get new direction of v_3

$\xi_6 : \begin{bmatrix} -\omega_6 \times q_\omega \\ \omega_6 \end{bmatrix}$

$$\omega_6' = e^{\hat{\omega}_1 \theta_1} e^{\hat{\omega}_2 \theta_2} e^{\hat{\omega}_4 \theta_4} e^{\hat{\omega}_5 \theta_5} \omega_6$$

↓

New direction only affected by revolute joints

ξ_3 doesn't affect it

$$q_\omega' = e^{\hat{\xi}_1 \theta_1} e^{\hat{\xi}_2 \theta_2} e^{\hat{\xi}_3 \theta_3} q_\omega$$

↓

ξ_4, ξ_5 pass through q_ω

Don't affect its location

ξ's to transform location of q_ω

$$J^b_{st}(\theta) = Ad^{-1}_{g_{st}} J^s_{st}(\theta)$$

$$\underline{V^s_{st}} = \underline{J^s_{st}} \cdot \dot{\theta}$$

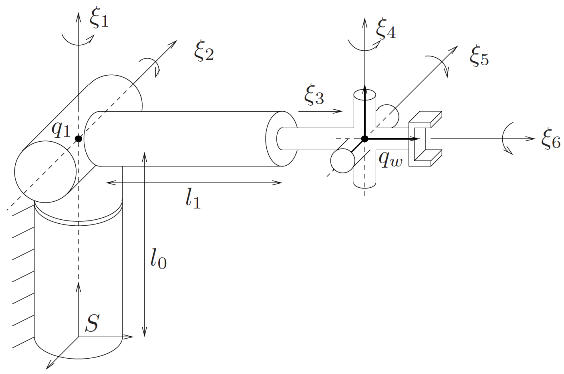
speeds

↓

New directions of joints

When we want to find the manipulator Jacobians for some specific configuration θ_d, it's easier to do it by inspection rather than having to first find the manipulator Jacobians for general θ, then plugging in θ_d. To find cross products, it may be helpful to draw out circles to visualize direction.

Problem 4. *Find the spatial and body manipulator Jacobians for the Stanford manipulator in its initial configuration. In this case, $\theta_d = 0$.*



- In reference config
- No joints have moved
- $\xi_i' = \xi_i$

$$J^S_{St}(0) = \begin{bmatrix} \xi_1 & \cdots & \xi_6 \end{bmatrix}$$

* Can use circle trick to calculate twists!
  ↳ Twists as velocities

5 Singularities

$$V_{st}^s = J_{st}^s(\theta)\dot{\theta}$$

At some configuration θ_s, it may be possible for $J_{st}^s(\theta_s)$ to *not* have full rank. In this case, $J_{st}^s(\theta_s)$ is not invertible, and thus the manipulator is unable to achieve instantaneous motion in certain directions. We call θ_s a *singular configuration.* Since being in singular configurations is not desirable, it's important to figure out what they are for a particular manipulator so they can be avoided.

Problem 5. *Show that a manipulator Jacobian is singular if there exist four revolute joint axes that intersect.*

Problem 6. *When is the elbow manipulator in a singular configuration?* → e.g. In reference config.

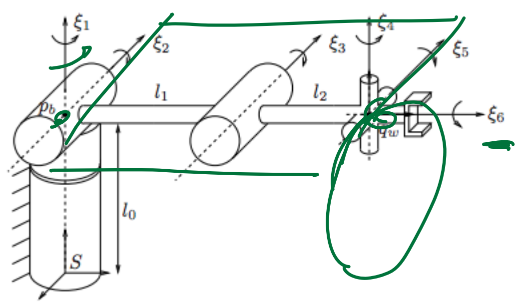


Figure 2: Elbow manipulator

- In some configurations, can't get a desired spatial velocity

$$V_{st}^{S} = \underbrace{J_{st}^{S}}_{\mathbb{R}^{6\times 6}} \cdot \dot{\Theta}$$

↓

< 6 linearly independent cols. → nonzero nullspace

In general, if rank $<$ min$(n, 6)$