EE106A Discussion 8: Jacobians

{isabella.huang, valmik}@berkeley.edu

1 Overview

Last week, we learned about spatial and body velocity twists between two frames A and B. These velocity twists are useful because they allow us to find the instantaneous velocity of the B frame expressed in both spatial and body coordinates.

$$v_{q_a}(t) \coloneqq \dot{q}_a(t) = \dot{g}_{ab}(t) q_b = \underbrace{\dot{g}_{ab}(t) g_{ab}^{-1}(t)}_{\coloneqq \widehat{V}_{ab}^s} q_a = \widehat{V}_{ab}^s q_a \tag{1}$$

$$v_{q_b}(t) \coloneqq g_{ab}^{-1}(t) v_{q_a}(t) = \underbrace{g_{ab}^{-1}(t) \dot{g}_{ab}}_{\coloneqq \widehat{V}_{ab}^b} q_b = \widehat{V}_{ab}^b q_b \tag{2}$$

Today, we will be thinking of velocities in the context of robotic manipulators. We will be finding the velocities between the fixed frame S and the end effector frame T, $\widehat{V}_{st}^s$ and $\widehat{V}_{st}^b$.

To do so, we will introduce the notion of spatial and body manipulator Jacobians. Then, we will see how these manipulator Jacobians help us detect singular configurations.

2 Adjoint for Twist Coordinate Change

When working with twists, we can transform a twist matrix $\widehat{\xi}$ into a different coordinate system defined by g, so that it becomes $\widehat{\xi}'$

$$\widehat{\xi}' = g \widehat{\xi} g^{-1} \tag{3}$$

In twist coordinates,

$$\xi' = Ad_g \xi \tag{4}$$

3 Spatial Jacobians

3.1 Definition

As before, we have the expression for $\widehat{V}_{st}^s$ as a function of the transformation between S and T:

$$\widehat{V}_{st}^s = \dot{g}_{st}(\theta)g_{st}^{-1}(\theta) \tag{5}$$

In twist coordinates,

$$V_{st}^s = J_{st}^s(\theta)\dot{\theta} \tag{6}$$

where the *spatial manipulator Jacobian* $J_{st}^s(\theta)$ is defined as

$$J_{st}^s(\theta) = \begin{bmatrix} (\frac{\partial g_{st}}{\partial \theta_1})^\vee & \dots & (\frac{\partial g_{st}}{\partial \theta_n})^\vee \end{bmatrix} \tag{7}$$

$$= \begin{bmatrix} \xi_1 & \xi_2' & \dots & \xi_n' \end{bmatrix} \tag{8}$$

$$\xi_i' = Ad_{(e^{\widehat{\xi}_1\theta_1} \dots e^{\widehat{\xi}_{i-1}\theta_{i-1}})}\xi_i \tag{9}$$

3.2 Interpretation

For some configuration θ, the spatial manipulator Jacobian maps the joint velocity vector $\dot{\theta}$ into the spatial velocity twist coordinates of the end-effector.

The i^{th} column of the spatial Jacobian ξ_i' is equal to the i^{th} joint twist transformed to the current manipulator configuration and written in spatial coordinates.

Problem 1. *Explain how this physical interpretation is true.*

3.3 How it's used

We can use the spatial Jacobian to compute the instantaneous velocity of a point q attached to the end-effector relative to the spatial frame. This velocity is

$$v_{q_s} = \widehat{V}_{st}^s q_s = (J_{st}^s(\theta)\dot{\theta})^\wedge q_s \tag{10}$$

where q_s is the coordinates of q in the spatial frame.

4 Body Jacobians

4.1 Definition

Now let's look at velocity twists in the body frame rather than in the spatial frame:

$$\widehat{V}^b_{st} = g^{-1}_{st}(\theta)\dot{g}_{st}(\theta) \tag{11}$$

In twist coordinates,

$$V^b_{st} = J^b_{st}(\theta)\dot{\theta} \tag{12}$$

where the *body manipulator Jacobian* $J^b_{st}(\theta)$ is defined as

$$J^b_{st}(\theta) = \begin{bmatrix} \xi^\dagger_1 & \xi^\dagger_2 & ... & \xi^\dagger_n \end{bmatrix} \tag{13}$$

$$\xi^\dagger_i = Ad^{-1}_{(e^{\widehat{\xi}_{i+1}\theta_{i+1}} ... e^{\widehat{\xi}_n\theta_n} g_{st}(0))}\xi_i \tag{14}$$

4.2 Interpretation

For some configuration θ, the body manipulator Jacobian maps the joint velocity vector $\dot{\theta}$ into the body velocity twist coordinates of the end-effector.

The i^{th} column of the body Jacobian $\xi^\dagger_i$ is equal to the i^{th} joint twist transformed to the current manipulator configuration and written in body coordinates.

Problem 2. *Explain how this physical interpretation is true.*

4.3 How it's used

We can use the body Jacobian to compute the instantaneous velocity of a point q attached to the end-effector relative to the body frame. This velocity is

$$v_{q_b} = \widehat{V}^b_{st} q_b = (J^b_{st}(\theta)\dot{\theta})^\wedge q_b \tag{15}$$

where q_b is the coordinates of q in the tool frame.

4.4 Converting between Spatial and Body Jacobians

$$J^s_{st}(\theta) = Ad_{g_{st}(\theta)} J^b_{st}(\theta) \tag{16}$$

Problem 3. *Find the spatial and body manipulator Jacobians for the Stanford manipulator.*

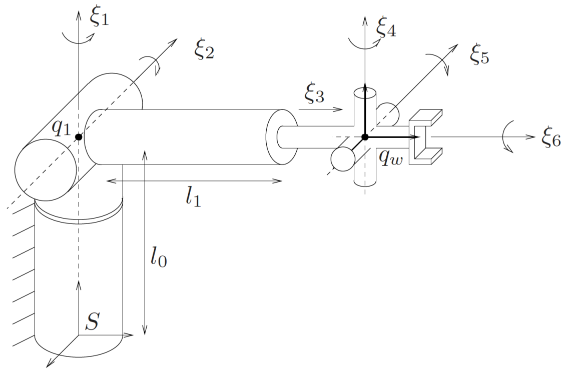
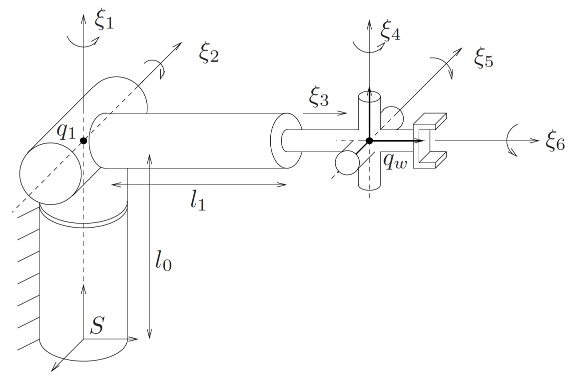


Figure 1: Stanford manipulator

When we want to find the manipulator Jacobians for some specific configuration θ_d, it's easier to do it by inspection rather than having to first find the manipulator Jacobians for general θ, then plugging in θ_d. To find cross products, it may be helpful to draw out circles to visualize direction.

Problem 4. *Find the spatial and body manipulator Jacobians for the Stanford manipulator in its initial configuration. In this case,* $\theta_d = 0$.



5 Singularities

$$V_{st}^s = J_{st}^s(\theta)\dot{\theta}$$

At some configuration θ_s, it may be possible for $J_{st}^s(\theta_s)$ to *not* have full rank. In this case, $J_{st}^s(\theta_s)$ is not invertible, and thus the manipulator is unable to achieve instantaneous motion in certain directions. We call θ_s a *singular configuration.* Since being in singular configurations is not desirable, it's important to figure out what they are for a particular manipulator so they can be avoided.

Problem 5. *Show that a manipulator Jacobian is singular if there exist four revolute joint axes that intersect.*

Problem 6. *When is the elbow manipulator in a singular configuration?*

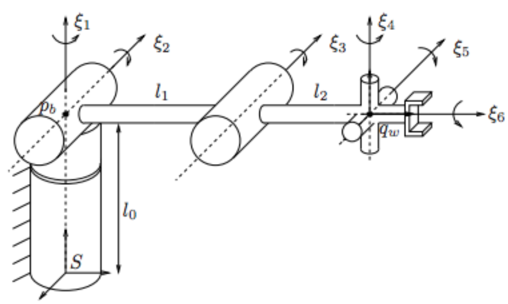


Figure 2: Elbow manipulator