EE106A Discussion 4: Inverse Kinematics

1 Inverse kinematics

In forward kinematics, we found the expression for $g_{st}(\theta)$ as a function of θ. Now, in inverse kinematics, we are given a desired configuration of the tool frame g_d, and we wish to find the θ for which

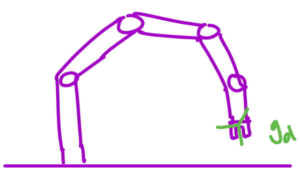
$$e^{\hat{\xi}_1\theta_1} \dots e^{\hat{\xi}_n\theta_n} g_{st}(0) = g_{st}(\theta) = g_d \tag{1}$$

Find

given

Given:

- Desired configuration
    - We know where we want our **tool** to end up
    - Ex. In position to grab a box on the table



$g_d \in SE(3)$

- Also know details about the robot itself
    - I.e., we know the **twists** and **starting configuration**

$\xi_1 \dots \xi_n$ $g_{st}(0)$

Desired:

- How do we angle each individual joint to get us there?
    - Allow us to move the robot to position it properly
    - Find **thetas**

$\theta_1 \dots \theta_n$

2 Padan-Kahan subproblems

To solve the inverse kinematics problem, one technique is to distill it into the following three simpler subproblems for which we know the solutions.

- We know the solutions to some **basic** inverse kinematics problems
    - If our problem is in the from of one of these basic ones, we can find theta
- Can we reduce the super complicated robot problem down to the basic ones?

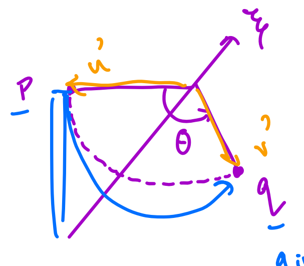
* Only revolute axes

Subproblems Overview

- Subproblem 1
  - Rotate about some fixed axis
  - Pure rotation about axis

$\xi = \begin{bmatrix} v \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ w \end{bmatrix}$

Pure rotation



rotation about w

$e^{\hat{\xi}\theta} p = q$

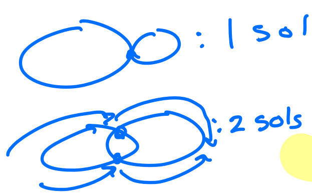
≤ 1 solution

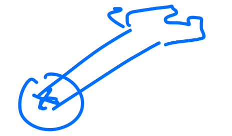
Need to specify (p, q)

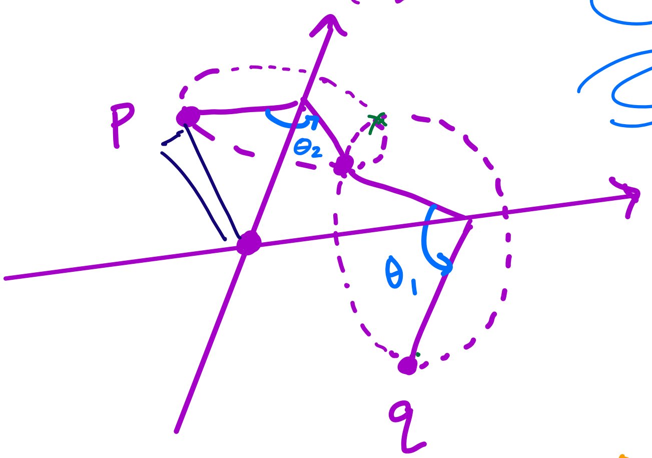
$\theta = \text{atan2}(w^T(u' \times v'), u' \cdot v')$

(won't need to ever compute this)

- Subproblem 2
  - Rotate about 2 intersecting axes







$e^{\hat{\xi}_1\theta_1} e^{\hat{\xi}_2\theta_2} p = q$

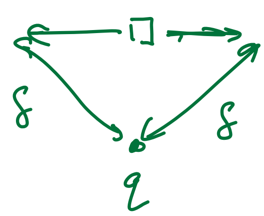
≤ 2 solutions

Specify (p, q)

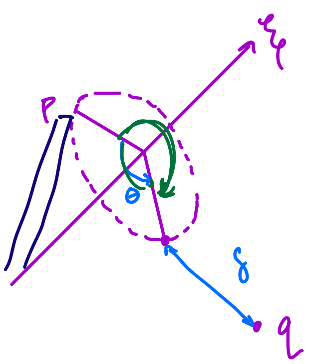
θ_1, θ_2 = * See discussion worksheet

- Subproblem 3
  - Move one point to a specified distance from another

*Apply to prismatic as well



$\| e^{\hat{\xi}\theta} p - q \| = \delta$



≤ 2 solutions

θ = * See worksheet solutions

Okay, so we know we can solve these subproblems. How does that help me with a large robot?

Great question. Our goal is to try to reduce the number of unknowns.

- Use specially chosen points
- Reduce the problem to only 1 or 2 unknown thetas
- Apply subproblems to solve for remaining variables

Tricks

$$e^{\hat{\xi}_1\theta_1} \cdots e^{\hat{\xi}_n\theta_n} g_{st}(0) = g_d$$

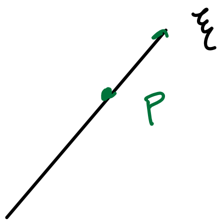
$\downarrow$

Rearrange

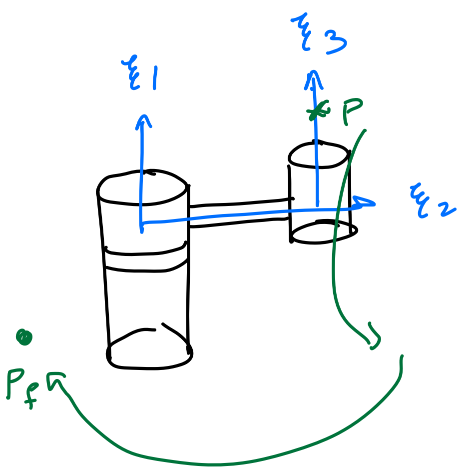
$\downarrow$

$$e^{\hat{\xi}_1\theta_1} \cdots e^{\hat{\xi}_n\theta_n} = g_d \, g_{st}^{-1}(0) := g$$

Trick #1: Choose a clever point (eliminate variables from RHS)



$$e^{\hat{\xi}\theta} p = p$$



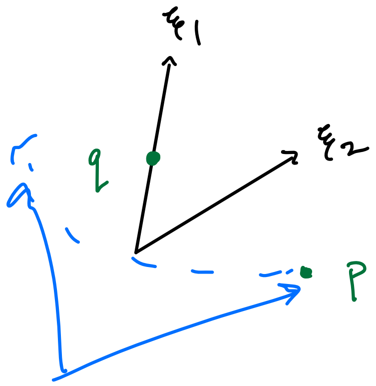
$$e^{\hat{\xi}_1\theta_1} e^{\hat{\xi}_2\theta_2} \underbrace{e^{\hat{\xi}_3\theta_3} \cdot p}_{\downarrow} = \underline{g \cdot p}$$

$$e^{\hat{\xi}_1\theta_1} e^{\hat{\xi}_2\theta_2} p = g \cdot p$$

$\longrightarrow$ Subproblem 2

Solve θ_1, θ_2

Trick 2: Subtract a point from both sides and take norm (eliminate variables from LHS)



$$e^{\hat{\xi}_1\theta_1} e^{\hat{\xi}_2\theta_2} = g$$

- q on axis of joint 1
- p not on axis of joint 2

q is on ξ_1

$$\| e^{\hat{\xi}_1\theta_1} e^{\hat{\xi}_2\theta_2} \cdot p - q \| = \| g \cdot p - q \|$$

$$\| \underbrace{e^{\hat{\xi}_1\theta_1}}_{\text{rigid body transform}} \underbrace{(e^{\hat{\xi}_2\theta_2} \cdot p - q)}_{\text{Vector}} \| = \| gp - q \|$$

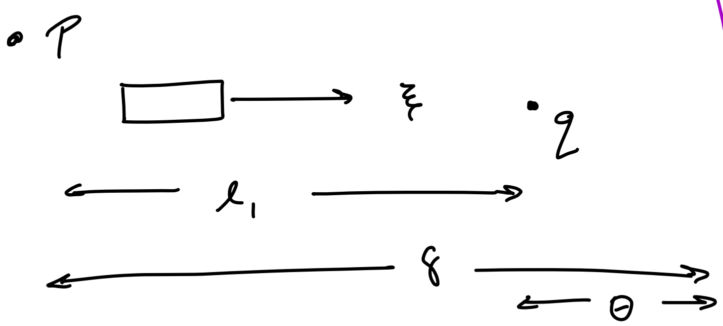
Transform won't change magnitude

$$\| e^{\hat{\xi}_2\theta_2} \cdot p - q \| = \| gp - q \|$$

→ Subproblem 3
Find θ_2

Trick 3: Prismatic Joints

- Good to solve these 1st
- Can sometimes be done by inspection (problem 2 of disc)
- Otherwise, use Subproblem 3



Get to this form w/ trick 2:

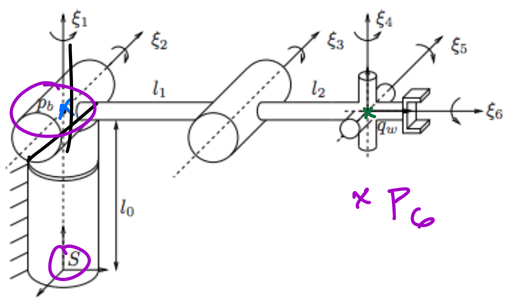
$$\| e^{\hat{\xi}\theta} p - q \| = \delta$$

$$\delta = l_1 + \theta$$

$$\theta = \delta - l_1$$

4 Elbow manipulator example

Break down the inverse kinematics for the elbow manipulator in Fig. 4 into simpler PK subproblems.



- Eliminating RHS: multiply
- Eliminating LHS: subtract

Figure 4: Elbow manipulator.

$$\underbrace{e^{\hat{\xi}_1\theta_1} e^{\hat{\xi}_2\theta_2} \cdots e^{\hat{\xi}_6\theta_6}}_{} \cdot \underline{g_{st}(0)} = g_d$$

$$e^{\hat{\xi}_1\theta_1} \cdots e^{\hat{\xi}_6\theta_6} = g_d \cdot g_{st}^{-1}(0) := g$$

Step 1: Use q_w

$$e^{\hat{\xi}_1\theta_1} \cdots e^{\hat{\xi}_6\theta_6} \cdot q_w = g \cdot q_w$$

q_w on ξ_4, ξ_5, ξ_6

$$\rightarrow \underline{e^{\hat{\xi}_1\theta_1} e^{\hat{\xi}_2\theta_2} e^{\hat{\xi}_3\theta_3}} \cdot q_w = g \cdot q_w$$

Step 2: Use P_b

$$e^{\hat{\xi}_1\theta_1} e^{\hat{\xi}_2\theta_2} e^{\hat{\xi}_3\theta_3} q_w - P_b = g \cdot q_w - P_b$$

$$\hookrightarrow \left\| \underline{e^{\hat{\xi}_1\theta_1} e^{\hat{\xi}_2\theta_2}} \left(\underline{e^{\hat{\xi}_3\theta_3} q_w - P_b} \right) \right\| = \left\| g \cdot q_w - P_b \right\|$$

$$\hookrightarrow \left\| e^{\hat{\xi}_3\theta_3} q_w - p_b \right\| = \left\| g \cdot q_w - p_b \right\|$$

$$\boxed{\begin{array}{c}\rightarrow \text{Subproblem } 3 \\ \text{Find } \theta_3\end{array}}$$

$$\underbrace{\underbrace{e^{\hat{\xi}_1\theta_1} e^{\hat{\xi}_2\theta_2}} \underline{g_3}} \cdot e^{\hat{\xi}_4\theta_4} e^{\hat{\xi}_5\theta_5} e^{\hat{\xi}_6\theta_6} = g$$

q_w doesn't effect $\xi_{4,5,6}$

Step 3: Use q_w

$$e^{\hat{\xi}_1\theta_1} e^{\hat{\xi}_2\theta_2} \cdot \underbrace{g_3 \cdot q_w}_{\text{point}} = \underbrace{g \cdot q_w}_{\text{point}}$$

$$\boxed{\begin{array}{c}\rightarrow \text{Subproblem } 2 \\ \theta_1, \theta_2\end{array}}$$

$$e^{\hat{\xi}_4\theta_4} e^{\hat{\xi}_5\theta_5} e^{\hat{\xi}_6\theta_6} = g_3^{-1} g_2^{-1} g_1^{-1} g := g'$$

Step 4: Use p_b again

$\hookrightarrow$ on ξ_6

Not on ξ_4 or ξ_5

$$e^{\hat{\xi}_4 \theta_4} e^{\hat{\xi}_5 \theta_5} \cdot p_b = g' \cdot p_b$$

$\rightarrow$ Subproblem 2

θ_4, θ_5

$$e^{\hat{\xi}_6 \theta_6} = g_5^{-1} g_4^{-1} g' := g''$$

Step 5: Pick any pt. not on ξ_6

$$e^{\hat{\xi}_6 \theta_6} \cdot p_6 = g'' \cdot p_6$$

Subproblem 1

Solve for θ_6

5 SCARA manipulator example

Break down the the inverse kinematics for the SCARA manipulator in Fig. 5 into simpler PK subproblems.

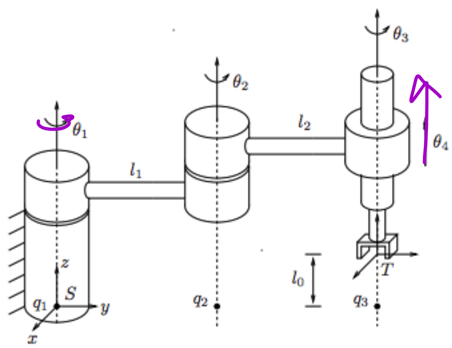


Figure 5: SCARA manipulator.

$$g_d = \begin{bmatrix} R & P \\ 0 & 1 \end{bmatrix}$$

$$P_z - l_0 = \theta_4$$