EECS/BioE/MechE C106A Discussion 2: Exponential Coordinates

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1 Rigid body transformations

Definition 2. A mapping $g : \mathbb{R}^3 \to \mathbb{R}^3$ is said to be a *rigid body transformation* if it satisfies the following properties:

1. Length is preserved: $\forall$ points $p, q \in \mathbb{R}^3$, $||p - q|| = ||g(p) - g(q)||$
2. The cross product (and therefore orientation) is preserved: $\forall$ vectors $v, w \in \mathbb{R}^3$, $g(v \times w) = g(v) \times g(w)$

Exercise. *Complete the mathematical statements for these two properties of rigid body transformations above.*

Proposition. A rotation matrix when applied as an operator via matrix multiplication is a valid rigid body transformation.

In general, rigid body transformations consist of rotation and translation as is depicted in Fig. 1.

1.1 Rigid transformation of a point

$$q_a = p_{ab} + R_{ab} q_b$$

So the rigid transformation g on a point q is

$$g(q) = p + R(q)$$

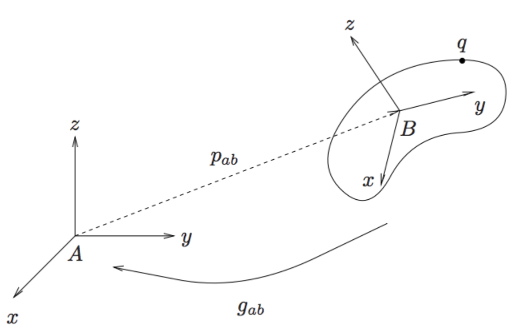

Figure 1: Illustration of a general rigid transformation between two frames.

This is an example of an *affine* transformation.

Definition 3. An affine transformation is a mapping $f : X \to Y$ of the form $f(x) = Mx + b$ where M is a linear transformation on the space X to the space Y and b is a vector in the space Y.

1.2 Rigid transformation of a vector

Given a vector $v = s - r$, we have the rigid transformation

$$g(v) = g(s - r) = g(s) - g(r) = R(s - r) = R(v)$$

so the rigid transformation of a vector consists of just a rotation.

1.3 Homogeneous coordinates

It not the most convenient to represent a rigid transformation by both a rotation matrix and a translation vector, so we can introduce homogeneous coordinates that helps simplify the representation. We now represent points with an extra 1 appended, and vectors with a 0 appended. For example, points and vectors in $\mathbb{R}^3$ are transformed to be in $\mathbb{R}^4$:

$$\bar{q} = \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ 1 \end{bmatrix} ; \quad \bar{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ 0 \end{bmatrix}$$

Now, a rigid transformation can be expressed in a purely *linear* transformation.

$$\bar{q}_a = \begin{bmatrix} q_a \\ 1 \end{bmatrix} = \begin{bmatrix} R_{ab} & p_{ab} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} q_b \\ 1 \end{bmatrix} =: \bar{g}_{ab}\bar{q}_b$$

$$\bar{g} = \begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix}$$

1.4 Composition Rule

As you might imagine, we can easily compose rigid body transforms given as 4×4 matrices together. The product of two rigid body transformations $g = g_1 g_2$ amounts to the transformation that takes in a point, applies the transform g_2 to it, and then applies the transform g_1 to the result.

When viewed as a change of coordinates between reference frames, rigid body transformations satisfy a similar composition rule to rotation matrices, and change of basis matrices in general.

$$g_{AC} = g_{AB} \cdot g_{BC} \tag{1}$$

1.5 Inverting a Rigid Body Transform

Rigid body transformations are always invertible. This should make intuitive sense. If any rigid body transformation can be thought of as a transform that converts the coordinates of a point from a reference frame A to another reference frame B, then of course we are capable of doing the opposite: recovering the point's coordinates in reference frame A given its coordinates in reference frame B. So we have that $g_{AB}^{-1} = g_{BA}$. In general, if a rigid transform g can be written as a 4×4 matrix of the form

$$g = \begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix}$$

then its inverse can be computed in closed form as

$$g^{-1} = \begin{bmatrix} R^T & -R^T p \\ 0 & 1 \end{bmatrix}$$

2 Exponential coordinates

In this class, we will repeatedly make use of *Exponential Coordinates* to describe rigid body transformations. Before talking about exactly what exponential coordinates are, we first present some linear algebra preliminaries that make it possible for us to talk about exponential coordinates.

2.1 The Matrix Exponential

Definition 1. The matrix exponential of A, e^A, is defined to be

$$e^A = I + A + \frac{A^2}{2!} + \frac{A^3}{3!} + ... \tag{2}$$

We are used to talking about the exponential function as a function on the reals $f : \mathbb{R} \to \mathbb{R}$ defined as $f(x) = e^x$. We can also write this function as an infinite series on the reals that converges everywhere:

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \tag{3}$$

We can use this infinite series to define the exponential as a function between matrices (!). Given a square matrix $A \in \mathbb{R}^{n \times n}$, define the *matrix exponential* of A as

$$e^A = \sum_{n=0}^{\infty} \frac{A^n}{n!} \tag{4}$$

This is well defined since we know how to take powers of matrices. Additionally, we also define $A^0 = I$ as the identity matrix for any square matrix A. The main property of the matrix exponential that we will find useful is in its application to solving linear vector differential equations.

Say we wish to find a vector trajectory $x(t) \in \mathbb{R}^n$ that satisfies the following vector differential equation

$$\dot{x} = Ax \tag{5}$$

for some constant $A \in \mathbb{R}^{n \times n}$ with initial condition $x(0) = x_0$. Then the matrix exponential provides the *unique* solution to this differential equation

$$x(t) = e^{At} x_0 \tag{6}$$

This gives us an easy way of finding the trajectory of a point in closed form if we can write its velocity as a linear function of its position. We will make extensive use of this property to derive the exponential coordinates representation of rigid body motion.

2.1.1 Exercise

1. By differentiating the series representation, show that if $Y(t) = e^{At}$ then $\dot{Y}(t) = Ae^{At} = e^{At}A$.
2. By differentiating the function $y(t) = e^{-At}x(t)$, show that $x(t) = e^{At}x_0$ is the unique solution to $\dot{x} = Ax$ with initial condition $x(0) = x_0$.

1. The series representaiton of e^{At} can be written out as the summation

$$e^{At} = \sum_{n=0}^{\infty} \frac{A^n t^n}{n!}$$

Taking the derivative of both sides yields

$$\begin{aligned} Ae^{At} &= \sum_{n=1}^{\infty} \frac{nA^n t^{n-1}}{n!} \\ &= (\sum_{n=0}^{\infty} \frac{A^n t^n}{n!})A \\ &= e^{At}A \end{aligned}$$

2. Consider the time derivative $\dot{y}$. Using the chain rule, we can write

$$\begin{aligned} \dot{y}(t) &= e^{-At}\dot{x}(t) - Ae^{-At}x(t) \\ &= e^{-At}Ax(t) - Ae^{-At}x(t) \\ &= (e^{-At}A - Ae^{-At})x(t) \\ &= 0 \end{aligned}$$

since $\dot{y}$ is uniformly 0 and y is clearly differentiable, y must be some constant vector $c \in \mathbb{R}^n$. So we have $e^{-At}x(t) = c \to x(t) = e^{At}c$. By substituting $t = 0$, it is clear that $c = x(0) = x_0$, giving us the unique solution for $x(t) = e^{At}x_0$ as needed.

2.2 Exponential coordinates for rotation

Problem 1. *Find the rotation matrix $R(\omega, \theta)$ for a rotation about some axis ω by amount θ. How is Rodrigues' formula related?*

In this formulation, ω is a vector that passes through the origin. We let point p be a point that rotates about ω with angular unit velocity, so $||\omega|| = 1$.

The velocity of this point is

$$\begin{aligned} \dot{p}(t) &= \omega \times p(t) \\ &= \widehat{\omega} p(t) \end{aligned}$$

Solving for this linear differential equation, we have

$$p(t) = e^{\widehat{\omega} t} p(0)$$

where $p(0)$ is the initial position of the point.

Since we constructed the angular velocity to be of unit magnitude, we can reparameterize by replacing t with θ in the solution, so

$$p(\theta) = e^{\widehat{\omega}\theta} p(0)$$

So, our desired rotation matrix is

$$R(\omega, \theta) = e^{\widehat{\omega}\theta}$$

Recall that Rodrigues' formula gives us a simpler way to calculate this matrix exponential:

$$e^{\widehat{\omega}\theta} = I + \widehat{\omega} sin\theta + \widehat{\omega}^2 (1 - cos\theta)$$

2.2.1 Exercise

Find the exponential coordinates of the following rotation matrices:

1. $R_x(\pi/2)$, the Euler x rotation matrix.
2. $R_y(-\pi/2)$
3. $R = R_x(\pi/2) R_y(-\pi)$

1. $\omega = (1, 0, 0)$, $\theta = \pi/2$.
2. $\omega = (0, 1, 0)$, $\theta = -\pi/2$.
3. If we draw out the initial and final frames, we realize that we seek a transformation that swaps the y and z axes, and flips the x axis to point in the opposite direction. The flipping of the x axis can be achieved by a rotation of π about any axis in the $y - z$ plane (perpendicular to x). Further, it is clear that a rotation about the axis $(0, 1, 1)$ will also swap the z and y axes, as needed. So we normalize and write the final answer as $\omega = (0, 1/\sqrt{2}, 1/\sqrt{2})$, $\theta = \pi$.

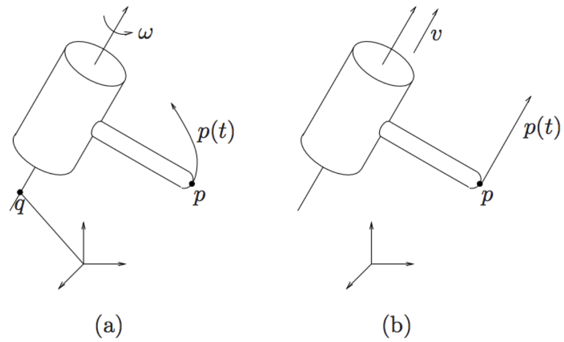

Figure 2: a) A revolute joint and b) a prismatic joint.

3 Exponential coordinates for rigid motion

Robotic rigid body transformations are enabled by *joints*, which connect sets of rigid *links* together. In this class, we focus on two types of joints — *revolute* and *prismatic* joints (see Fig. 2). Revolute joints allow adjacent links to rotate relative to each other about a fixed axis, and prismatic joints allow links to move linearly relative to each other along a fixed axis.

Problem 2. *Write the expressions for the velocity of the point p (ie. $\dot{p}(t)$) when attached to both the revolute and prismatic joints in Fig. 2. Assume that $\omega \in \mathbb{R}^3$, $||\omega|| = 1$, and $q \in \mathbb{R}^3$ is some point along the axis of ω.*

For the revolute joint: $\dot{p}(t) = \omega \times (p(t) - q)$

For the prismatic joint: $\dot{p}(t) = v$

3.1 Twist of revolute joint

Let's transform the above expression for velocity into homogeneous coordinates. Recall that in homogeneous coordinates, append a 0 to vectors and a 1 to points.

Problem 3. *Find $\widehat{\xi}$ to complete the following expression of $\dot{p}(t)$ in homogeneous coordinates for a revolute joint.*

$$\begin{bmatrix} \dot{p} \\ 0 \end{bmatrix} = \underbrace{\begin{bmatrix} \widehat{\omega} & -\omega \times q \\ 0 & 0 \end{bmatrix}}_{=:\,\widehat{\xi}} \begin{bmatrix} p \\ 1 \end{bmatrix}$$

Hint: Recall the skew symmetric matrix $\widehat{w}$ of w:

$$\omega = [\omega_1 \;\; \omega_2 \;\; \omega_3]^T; \;\; \widehat{w} = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix} \tag{7}$$

3.2 Twist of prismatic joint

Problem 4. *Find $\widehat{\xi}$ to complete the following expression of $\dot{p}(t)$ in homogeneous coordinates for a prismatic joint.*

$$\begin{bmatrix} \dot{p} \\ 0 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & v \\ 0 & 0 \end{bmatrix}}_{=:\, \widehat{\xi}} \begin{bmatrix} p \\ 1 \end{bmatrix}$$

3.3 Vee and wedge operators of a twist

The above quantity we have derived, $\widehat{\xi}$, is called a *twist*. A twist captures the angular and linear velocities of a body. There are two handy operators that we use on twist entities. First, given a twist $\widehat{\xi} = \begin{bmatrix} \widehat{\omega} & v \\ 0 & 0 \end{bmatrix} \in \mathbb{R}^4$, the $\vee$ (vee) operator extracts the 6-dimensional vector which parameterizes a twist, where $\xi \coloneqq (v, \omega)$ are the *twist coordinates* of $\widehat{\xi}$.

$$\begin{bmatrix} \widehat{\omega} & v \\ 0 & 0 \end{bmatrix}^{\vee} = \begin{bmatrix} v \\ \omega \end{bmatrix} =: \xi \tag{8}$$

The inverse operator, $\wedge$ (wedge), constructs a matrix out of a vector of the twist coordinates:

$$\begin{bmatrix} v \\ \omega \end{bmatrix}^{\wedge} = \begin{bmatrix} \widehat{\omega} & v \\ 0 & 0 \end{bmatrix} \tag{9}$$

3.4 Solution to differential equation gives us the exponential map

Problem 5. *Write the general solution to the differential equation $\dot{\bar{p}} = \widehat{\xi}\bar{p}$. Then, make use of the fact that $||\omega|| = 1$ to reparameterize t to be θ. Specifically, find the expression for $p(\theta)$ in terms of $p(0)$.*

$$\bar{p}(t) = e^{\widehat{\xi}t}\bar{p}(0)$$

$$\bar{p}(\theta) = e^{\widehat{\xi}\theta}\bar{p}(0)$$

This transformation is not the same as the rigid transformations we studied previously in that it is not a mapping from one coordinate frame to another, but rather the mapping of points from their initial coordinates $p(0)$ to their coordinates after a rigid motion parameterized by a joint angle θ is applied. It turns out that the matrix exponential simplifies to:

$$e^{\hat{\xi}\theta} = \begin{cases} \begin{bmatrix} I & v\theta \\ 0 & 1 \end{bmatrix} & \omega = 0 \\ \\ \begin{bmatrix} e^{\hat{\omega}\theta} & \left(I - e^{\hat{\omega}\theta}\right)(\omega \times v) + \omega\omega^T v\theta \\ 0 & 1 \end{bmatrix} & \omega \neq 0,\ ||\omega|| = 1 \end{cases} \tag{10}$$

4 Screw motion

Chasles' theorem states that any rigid body transformation can be decomposed into an equivalent finite rotation about a fixed axis and a finite translation along that axis. This is what we call a *screw motion* S, which consists of an axis l, a pitch h, and a magnitude M. It is equivalent to a rotation by an amount $\theta = M$ about l followed by a translation by $h\theta = hM$ along l (see Fig. 4). ($h = 0$ corresponds to pure rotation, and $h = \infty$ corresponds to pure translation).

The transformation g corresponding to S has the following effect on a point p:

$$gp = q + e^{\hat{\omega}\theta}(p - q) + h\theta\omega \tag{11}$$

Problem 6. *Convert this transformation to homogeneous coordinates. What do you notice between this expression and the one in Eq. 10?*

$$g \begin{bmatrix} p \\ 1 \end{bmatrix} = \begin{bmatrix} e^{\widehat{\omega}\theta} & (I - e^{\widehat{\omega}\theta})q + h\theta\omega \\ 0 & 1 \end{bmatrix} \begin{bmatrix} p \\ 1 \end{bmatrix}$$

This is very similar in form to the expression in Eq. 10. This is just to build intuition that every twist corresponds to an equivalent screw motion and vice versa.

5 Twists from Screw Motions

Every screw motion corresponds to a Twist. We consider separately the case where the given screw is a pure translation, ($h = \infty$) and when it is a nonzero rotation followed by some finite translation ($h < \infty$).

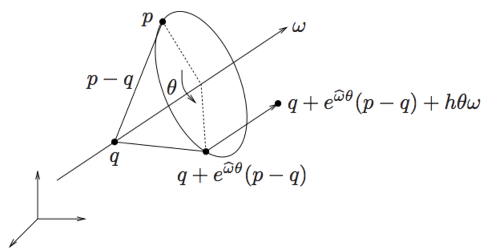

Figure 3: Generalized screw motion.

1. ($h = \infty$) If the given screw motion is a translation along a unit vector v by θ units, then the corresponding twist is

$$\hat{\xi} = \begin{bmatrix} v \\ 0 \end{bmatrix}^{\wedge} = \begin{bmatrix} 0 & v \\ 0 & 0 \end{bmatrix} \tag{12}$$

and the exponential coordinates of the generated rigid motion are (ξ, θ). In other words, the rigid body transformation produced by the given screw is $g = e^{\hat{\xi}\theta}$.

2. (h is finite) If the given screw motion is a rotation by θ about some axis in the direction of the unit vector ω that passes through a point q (so that the set of all points on the axis can be written as $\{p : p = q + \lambda\omega \text{ for some } \lambda \in \mathbb{R}\}$) followed by a translation along the same axis ω by $h\theta$ units, then the corresponding twist is

$$\hat{\xi} = \begin{bmatrix} -\omega \times q + h\omega \\ \omega \end{bmatrix}^{\wedge} = \begin{bmatrix} \hat{\omega} & -\omega \times q + h\omega \\ 0 & 0 \end{bmatrix} \tag{13}$$

and the exponential coordinates of the generated rigid motion are (ξ, θ). In other words, the rigid body transformation produced by the given screw is $g = e^{\hat{\xi}\theta}$.