Kinematics $\rightarrow$ positions

$$g_{AB}(\theta) = e^{\hat{\xi}\theta} g_{AB}(0) \quad , \quad \xi = \begin{bmatrix} v \\ w \end{bmatrix} = \begin{bmatrix} \text{linear} \\ \text{angular} \end{bmatrix}$$

Kinetics $\rightarrow$ velocities

$$\dot{q}_s = \hat{V}^s q_s \qquad V^s = \xi = \begin{bmatrix} v^s \\ w^s \end{bmatrix}$$

Dynamics $\rightarrow$ forces

$$\Gamma \in \mathbb{R}^{6\times 1} = \begin{bmatrix} \text{liner} \\ \text{angular} \end{bmatrix} = \begin{bmatrix} \text{force} \\ \text{torque} \end{bmatrix}$$

$\tau \hookrightarrow$ torque on some joint if we apply a wrench to our body coord frame

$$= \underbrace{\xi^{+T}}_{\text{current body twist transpose}} \cdot \Gamma^B$$

C106B Discussion 6: Grasping

1 Introduction

Grasping objects is a major part of current robotic manipulation research. To approach this problem, we will discuss wrenches and their mathematical properties and then apply them to the idea of contact forces.

2 Wrenches

Last semester, we started our discussion of robotic arm movement talking about *kinematics*. This deals with the different positions and angles our body frame can potentially reach. The orientation of the **B** frame with respect to the **A** frame is given by the forward kinematic map:

$$g_{AB}(\theta_1) = e^{\hat{\xi}_1 \theta_1} g_{AB}(0)$$

We then discussed *kinetics*, which deal with velocities and accelerations. The relative velocity of a point given in the body frame for some angular velocity $\dot{\theta}$ is

$$v_{q_S} = \hat{V}^s_{AB} q_S, \;\; V^s_{AB} = \xi' \dot{\theta} = \begin{bmatrix} v^s_{AB} \\ \omega^s_{AB} \end{bmatrix}$$

$$v_{q_B} = \hat{V}^b_{AB} q_B, \;\; V^b_{AB} = \xi^\dagger \dot{\theta} = \begin{bmatrix} v^b_{AB} \\ \omega^b_{AB} \end{bmatrix}$$

where ξ' and $\xi^\dagger$ are the current spatial and body twists.

Now, we move to dynamics for an arm - analyzing the relationship between *forces* applied on the body and its motion! A *wrench* follows the same kind of linear/angular form as twists:

$$\Gamma = \begin{bmatrix} f \\ \tau \end{bmatrix}$$

where f is a linear force component, and τ is a torque.

Unlike angular velocities, however, to compute the total torque on some joint, we use the *transpose* of the twist.

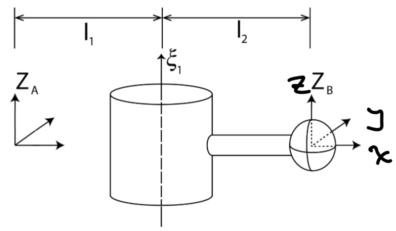
$$\tau = \xi'^T \Gamma^S$$

$$\tau = \xi^{\dagger^T} \Gamma^B$$

The difference between the spatial and body wrench Γs is the frame in which we are applying the wrench.

$$\tau_1 = \xi^T \Gamma^B$$

Problem 1: How does a wrench Γ^B applied on the **B** frame affect the torque at joint ξ_1?



$$\tau_1 = \xi^T \Gamma^B$$

$$\downarrow$$

$$= \begin{bmatrix} -w \times q \\ w \end{bmatrix}$$

Express ξ_1 in terms of body frame:

$$w = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \qquad q = \begin{bmatrix} -l_2 \\ 0 \\ 0 \end{bmatrix}$$

$$\xi = \begin{bmatrix} 0 \\ l_2 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \rightarrow \xi^T \Gamma^B = \tau$$

3 Adjoints for Wrenches

Spatial and body velocities are related to one another using adjoints (which are invertible):

$$V_{AB}^S = Ad_{g_{AB}} V_{AB}^b$$

$$Ad_g = \begin{bmatrix} R & \hat{p}R \\ 0 & R \end{bmatrix}$$

Can we figure out a similar relationship for wrenches?

It turns out we can!

$$\Gamma^S = Ad_{g_{AB}^{-1}}^T \Gamma^B$$

Problem 2: The work of a force is calculated by $W = F \cdot d$. Two wrenches are equivalent if they generate the same amount of work. Use this concept to prove the adjoint relationship for wrenches.

Rewrite work using velocities

$$W = \int_{t_1}^{t_2} \Gamma \cdot V \, dt$$

$$W = \int_{t_1}^{t_2} \Gamma^S \cdot V^S dt = \int_{t_1}^{t_2} \Gamma^B \cdot V^B \, dt$$

Infinitesimally small movement

$$\Gamma^S \cdot V^S = \Gamma^B \cdot V^B$$

$$(\Gamma^S)^T V^S = (\Gamma^B)^T V^B$$

Express V^B in the spatial frame

$$(\Gamma^S)^T V^S = (\Gamma^B)^T \cdot Ad_{g_{BS}} V^S$$

$$(\Gamma^S)^T = (\Gamma^B)^T Ad_{g_{BS}}$$

$$\Gamma^S = Ad_{g_{BS}}^T \, \Gamma^B$$

$$\boxed{\Gamma^S = Ad_{g_{SB}^{-1}}^T \, \Gamma^B}$$

4 Jacobians for Wrenches

Last semester, we used the spatial and body Jacobians to transform individual joint velocities to the end-effector velocity and vice-versa. You also used this concept in Project 1 to generate jointspace trajectory commands. Recall that

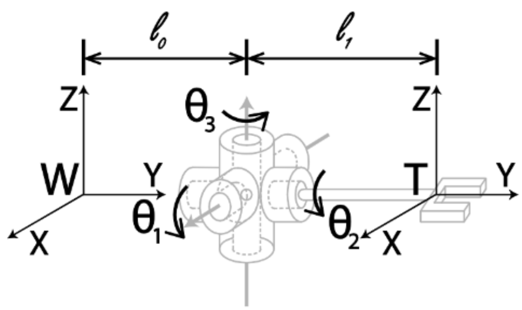
$$V^S = J^S(\theta)\dot{\theta}$$
$$V^B = J^B(\theta)\dot{\theta}$$

where $\dot{\theta}$ is a vector of individual joint velocities. The Jacobian itself is a composition of the individual joint twists in their current configuration.

To compute the torques on each joint based on a wrench applied in the spatial or body frame, we can use the Jacobian as well:

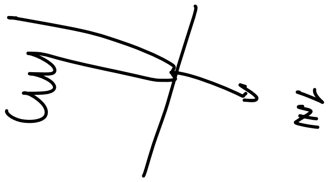
$$\begin{bmatrix} \tau_1 \\ \vdots \\ \tau_n \end{bmatrix} = (J^S)^T \Gamma^S = (J^B)^T \Gamma^B$$

Problem 3: Compute the joint torques if we apply a force on the body frame.



Contact points $\rightarrow$ positions where we grab object

ex. Where Sawyer gripper is holding object



Contact frame has z-axis pointing into contact

(x & y axes are $\perp$)

If I have friction, object is able to resist forces in $\perp$ directions

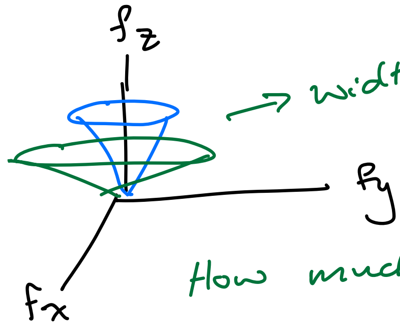
(x & y directions)

Can also resist a torque in the z-direction (assuming we use a finger / gripper instead of a single point)

B/c of friction, force in x & y directions that can be resisted is limited by μ:

$\hookrightarrow$ $$\sqrt{f_x^2 + f_y^2} \le \mu f_z$$

Normal force + coeff. of friction determine friction force



$\rightarrow$ width depends on μ

How much force can I resist w/ a contact?

$$|\tau_z| \leq \gamma f_z$$

f_x, f_y, f_z, & τ_z are resistable w/ a contact

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} f_x \\ f_y \\ f_z \\ \tau_z \end{bmatrix} \rightarrow \underbrace{\begin{bmatrix} \\ \\ \\ \\ \end{bmatrix}}_{\text{wrench}} \in \mathbb{R}^6$$

→ Use the adjoint to transform our grasp wrench into world frame

→ Multiple different contacts:

- Transform into world frame
- Sum up

⇓

wrenches that our grasp can resist

5 Grasp map

When going into robotic hands, we want to actually grab objects. One *contact* is defined by

$$F_{c_i} = B_{c_i} f_{c_i}$$

Where B is the contact basis, or the directions in which the contact can apply force, and f is a vector in that basis (the actual forces being applied). F is the 6x1 wrench which the contact applies. In our case, we use a soft contact model, which has both lateral and torsional friction components, so the basis is

$$B_{c_i} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

However, in the real world, friction is not infinite. For the contact to resist a wrench without slipping, the contact vector must lie within the *friction cone*, which is defined

$$FC_{c_i} = \{f \in \mathbb{R}^4 : \sqrt{f_1^2 + f_2^2} \leq \mu f_3, f_3 > 0, |f_4| \leq \gamma f_3\}$$

f_3 is the amount of normal force being applied, f_1 and f_2 are the forces in the other two perpendicular directions, and f_4 is a torque. The friction cone therefore tells us the forces that can be applied onto an object that would be resisted by this contact.

However, we want the wrenches that a contact point can resist in the world frame, not the contact frame. So we use the adjoint to transform the contact basis:

$$G_i := \begin{bmatrix} R_{oc_i} & 0 \\ \widehat{p}_{oc_i} R_{oc_i} & R_{oc_i} \end{bmatrix} B_{c_i} = Ad^T_{g^{-1}_{co_i}} B_{c_i}$$

A grasp is a *set* of contacts (maybe multiple fingers in a hand or the two sides of a Sawyer gripper), so we define the wrenches (in the world frame) a grasp can resist as:

$$F_o = G_1 f_{c_1} + \cdots + G_k f_{c_k} = \begin{bmatrix} G_1 & \cdots & G_k \end{bmatrix} \begin{bmatrix} f_{c_1} \\ \vdots \\ f_{c_k} \end{bmatrix} = Gf$$

The resulting compound matrix G above is called the *grasp map*, summing up multiple forces.

6 Force closure

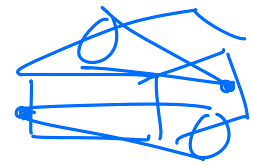
A grasp is in *force closure* when finger forces lying in the friction cones span the space of object wrenches

$$G(FC) = \mathbb{R}^6$$

Essentially, this means that any external wrench applied to the object can be countered by the sum of contact forces (provided the contact forces are high enough).

For a two-contact soft-fingered grasp, we also have the following theorem which makes it very easy to check when a grasp is in force closure. This is theorem 5.7 from MLS.

Theorem. *A spatial grasp with two soft-finger contacts is force-closure if and only if the line connecting the contact point lies inside both friction cones.*



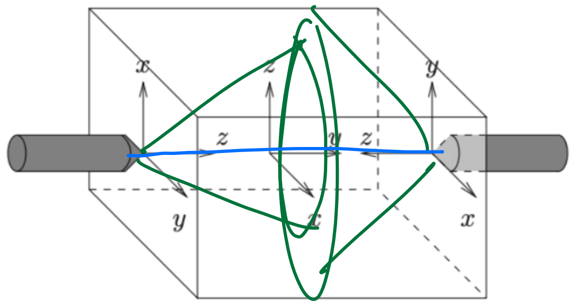


Figure 1: Two finger grasp.

6.1 Discretizing the Friction Cone

Checking that $f \in FC$ can be difficult. Often when evaluating grasps, we will write down an optimization problem that has $f \in FC$ as a constraint.

$$FC_{c_i} = \begin{cases} \sqrt{f_1^2 + f_2^2} \leq \mu f_3 \\ f_3 > 0 \\ |f_4| \leq \gamma f_3 \end{cases}$$

We can approximate the (conical) friction cone as a pyramid with n vertices. The level sets of the friction cone are circles, but the level sets for this convex approximation are n sided polygons circumscribed by the circle. Thus, the interior of this convexified friction cone is a conservative approximation of the friction cone itself.

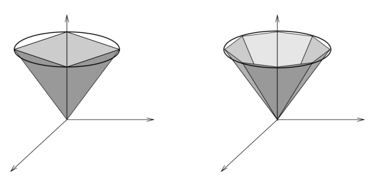


Figure 2: Approximations of the friction cone. From section 5.3 of MLS.

Any point in the interior of this pyramid can be described as a sum of

$$f = \alpha_0 f_0 + \sum_{i=1}^{n} \alpha_i f_i = F\alpha$$

where f_i are the edges of the pyramid and f_0 a straight line in z, and the weights α are all non-negative. Here, we can write a composite matrix F (different from the F above!) with the f_i vectors as its columns. This lets us more easily characterize any f in the friction cone. We make the approximation that $f \in FC$ if and only if there exists a non-negative vector α such that $f = F\alpha$.

With this approximation, the condition that $f \in FC$ is equivalent to the pair of linear constraints $\{f = F\alpha, \alpha \geq 0\}$ (where this inequality is understood to be element-wise).

Problem 4:
Let w be a given wrench. Let a two-contact grasp be given to you with contact grasp maps G_1 and G_2. We wish to find the input force $f \in FC$ with the smallest norm that can resist the wrench w applies at the center of mass of the object being grasped. Using the polyhedral approximation of the friction cone, write this as a quadratic program.

Optimization goal: Find the min. forces to apply to hold our object

$$\min \underbrace{f^T f}$$

$$\hookrightarrow f = \begin{bmatrix} f_a \\ f_b \end{bmatrix} = \text{forces from each contact}$$

Constraints:

- Positive normal force $f_{a,z} \quad f_{b,z} \geq 0$
- Want to make sure our forces lie in friction cone
  $f_a = F\alpha_a \quad f_b = F a_b$
- Want to best oppose external wrenches
  $-w = Gf$ → provide some opposing force

Problem 5:
Consider the box grasped by 2 soft-finger contacts shown in the figure above. Find the grasp map. Assume the object is a cube of side-length 2.