

C106B Discussion 3: Feedback Linearization

1 Introduction

Today, we'll talk about:

1. SISO Feedback Linearization
2. Relative Degree
3. MIMO Feedback Linearization

2 SISO Feedback Linearization

Linear systems, of the form $\dot{x} = Ax + Bu, y = Cx$, are the simplest form of dynamical system to analyze and control. Can we use an input, u , to make a control affine nonlinear system:

$$\dot{x} = f(x) + g(x)u \quad (1)$$

$$y = h(x) \quad (2)$$

Where $g(x) \neq 0$, behave like a linear system? Let's consider the single input, single output (SISO) case, where $u, y \in \mathbb{R}, x \in \mathbb{R}^n$. Let's try focusing on the output of the system, y , the variable we actually wish to control. Computing its derivative along the trajectories of the system:

$$\dot{y} = \frac{\partial h}{\partial x} f(x) + \frac{\partial h}{\partial x} g(x)u \quad (3)$$

The terms $\frac{\partial h}{\partial x} f(x)$ and $\frac{\partial h}{\partial x} g(x)$ are called **Lie derivatives** (pronounced "Lee" derivatives).

$$L_f h(x) = \begin{bmatrix} \frac{\partial h}{\partial x_1} & \dots & \frac{\partial h}{\partial x_n} \end{bmatrix} \begin{bmatrix} f_1(x) \\ \vdots \\ f_n(x) \end{bmatrix} \quad (4)$$

The Lie derivative of $h(x)$ along $f(x)$ tells us how quickly h changes along the vector field $f(x)$ - we can think of it like an inner product between $\frac{\partial h}{\partial x}$ and $f(x)$. Using Lie derivative notation:

$$\dot{y} = L_f h(x) + L_g h(x)u \quad (5)$$

Problem 1: Consider the SISO system above. Assuming $L_g h(x) \neq 0$ in our region of interest, find a formula for u such that when u is plugged into the output dynamics, the following equation results:

$$\dot{y} = v \quad (6)$$

Where $v \in \mathbb{R}$ is an arbitrary scalar we can control.

3 Relative Degree

What if $L_g h(x) = 0$ in the SISO case? We can try taking *higher* derivatives of y until $L_g h(x) \neq 0$ and our input term appears. The smallest level of derivative for which the input will appear in the derivative of y is called the **relative degree**, r , of the system.¹

Assuming that the input does not show up for all derivatives less than r , how can we express the r^{th} derivative of the output using Lie derivative notation?

$$y^{(r)} = L_f^r h(x) + L_g L_f^{r-1} h(x) u \quad (7)$$

By the definition of relative degree, at the r^{th} derivative, $L_g L_f^{r-1} h(x) \neq 0$ in our region of interest. This means that we can pick a feedback linearizing control law:

$$u = \frac{1}{L_g L_f^{r-1} h(x)} (-L_f^r h(x) + v) \Rightarrow y^{(r)} = v \quad (8)$$

Where v is an arbitrary scalar. Note that as a general rule of thumb for SISO systems, $r \leq n$ if $g(x) \neq 0$.

Problem 2: Consider a simple pendulum which swings under gravity with some friction. A torque, which we can control, is applied to the pendulum at its pivot point. The equations of motion of this system may be written as a control affine, SISO system in state space as:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ -\frac{g}{l} \sin x_1 - \beta x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \quad (9)$$

$$y = x_1 \quad (10)$$

Where $[x_1, x_2] = [\theta, \dot{\theta}]$ and $u = \tau$, the torque applied to the pendulum. The output of this system is $y = x_1$, the angle of the pendulum. Take the time derivatives of the output along the trajectories of the system until the input, u , appears. What is the relative degree of the system? *Hint: Here, it's easier to take the time derivative directly instead of using Lie derivative notation.*

Problem 3: Find a feedback linearizing input u to the pendulum system such that when the input is applied to the system, the following differential equation:

$$\ddot{y} = v \quad (11)$$

Where $v \in \mathbb{R}$ is an arbitrary scalar, governs the dynamics.

¹Note that relative degree is actually defined at a particular point in the domain.

4 MIMO Linearization

How can we generalize our feedback linearization results to a multi input, multi output (MIMO) system? We will consider *square* MIMO systems, which have the same number of inputs as outputs:

$$\dot{x} = f(x) + g(x)u, \quad x \in \mathbb{R}^n, u \in \mathbb{R}^m \quad (12)$$

$$y = h(x), \quad y \in \mathbb{R}^m \quad (13)$$

MIMO feedback linearization will *largely* be the same as SISO feedback linearization! We can think about the output y , as a collection of single outputs:

$$y = \begin{bmatrix} y_1 \\ \vdots \\ y_m \end{bmatrix} = \begin{bmatrix} h_1(x) \\ \vdots \\ h_m(x) \end{bmatrix} \quad (14)$$

How can we express the time derivatives of these outputs using Lie derivative notation? In this case, instead of being a vector, $g(x)$ will be a matrix:

$$g(x) = \begin{bmatrix} | & & | \\ g_1(x) & \dots & g_m(x) \\ | & & | \end{bmatrix} \quad (15)$$

Problem 4: Show that the first time derivative of the output y_j along the trajectories of the system is computed:

$$\dot{y}_j = L_f h_j(x) + \sum_{i=1}^m L_{g_i} h_j(x) u_i \quad (16)$$

Hint: Begin by applying the chain rule, try multiplying $\frac{\partial h_j}{\partial x}$ by each column of $g(x)$ to get the sum!