C106B/206B Discussion 2: Lyapunov Stability and Optimization

1 Introduction

Today, we'll talk about:

1. Lyapunov Stability
2. Energy-Like Functions (LPDF, PDF)
3. The Basic Theorem of Lyapunov
4. Basics of Optimization

2 Lyapunov Stability

What does it mean for an equilibrium point to be stable? Recall that last week, we discussed what an intuitive definition of stability should be - a stable equilibrium point should be such that if we *start* close to the point, we'll *stay* close to the point for all time!

We can formalize this concept using the definition of stability in the sense of Lyapunov (SISL), or Lyapunov stability for short! Note that if $x_e \neq 0$ but we wish to study it, we may simply perform a change of coordiantes that shifts x_e to be the zero vector.

Definition 1 ***Stability in the Sense of Lyapunov (SISL)***
The equilibrium point $x_e = 0$ of the nonlinear system:

$$\dot{x} = f(x, t),\ x \in \mathbb{R}^n,\ t \in \mathbb{R}^+ \tag{1}$$

Is said to be stable in the sense of Lyapunov at $t = t_0$ if for all $\varepsilon > 0$, there exists a $\delta(t_0, \varepsilon) > 0$ such that $||x(t_0)|| < \delta$ implies:

$$||x(t)|| < \varepsilon\ \forall t \geq t_0 \tag{2}$$

This definition is **local**, and not too restrictive! There are some stronger versions that ensure convergence and rates of convergence to the equilibrium point.

At the moment, to apply the definition of Lyapunov stability, we'll need to find an explicit solution to the nonlinear differential equation, $\dot{x} = f(x, t)$! This is often challenging and in many cases impossible to write in a closed form.

3 Energy-Like Functions

For physical systems, we can think about studying the energy of a system to determine stability. If energy is always or almost always decreasing with respect to time for some region, we can make conclusions about the stability of the equilibrium points of a physical system!

Not all systems can be described using methods of dynamics! Can we define a measure of energy for a *arbitrary* nonlinear system? We have two possible types of "energy-like functions," one with stricter conditions than the other!

Definition 2 ***Locally Positive Definite Function (LPDF)***
A function $V(x,t) : \mathbb{R}^n \times \mathbb{R}^+ \to \mathbb{R}^+$ is a locally positive definite function if there exists a continuous, strictly increasing function $\alpha : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\alpha(0) = 0$ and:

1. $V(0,t) = 0$ *for all* $t \geq 0$
2. $V(x,t) \geq \alpha(||x||)$ *for all* x *such that* $||x|| < \varepsilon$ *and all* $t \geq 0$

Note that this is *not* the only definition of an LPDF! There is an equivalent more advanced definition that's sometimes easier to use as well!

The stronger version of an LPDF is a positive definite function, PDF, which extends the result to its entire domain!

Definition 3 ***Positive Definite Function (PDF)***
A function $V(x,t) : \mathbb{R}^n \times \mathbb{R}^+ \to \mathbb{R}^+$ is a positive definite function if there exists a continuous, strictly increasing function $\alpha : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\alpha(0) = 0$ and:

1. $V(0,t) = 0$ *for all* $t \in \mathbb{R}^+$
2. $V(x,t) \geq \alpha(||x||)$ *for all* $x \in \mathbb{R}^n$ *and all* $t \geq 0$
3. $\lim_{p\to\infty} \alpha(p) = \infty$

Problem 1: Suppose $P \in \mathbb{R}^{n\times n}$ is a diagonal matrix with all positive, real eigenvalues.

$$P = \begin{bmatrix} \lambda_1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \lambda_n \end{bmatrix} \tag{3}$$

Show that the function $V(x) = x^T P x$ is a positive definite function (PDF).

Solution: If $V(x) = x^T P x$, we may multiply out the matrix and vectors to get:

$$V(x) = \lambda_1 x_1^2 + ... + \lambda_n x_n^2 \geq \lambda_{min}||x||^2 \tag{4}$$

Since all of the eigenvalues are positive and real, $\lambda_{min}||x||^2$ is a strictly increasing function that satisfies the conditions in the definition above.

4 The Basic Theorem of Lyapunov

We now need a method to compute the time derivative of the Lyapunov function!

Definition 4 ***Derivative Along a Trajectory***
The first time derivative of the function $V(x,t) : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ along the trajectories of the system $\dot{x} = f(x,t)$ is defined:

$$\dot{V} = \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(x,t) \tag{5}$$

Let's apply the idea of "rate of change of energy" to study stability.

Theorem 1 ***Basic Theorem of Lyapunov (Direct Method)***
Suppose $\dot{x} = f(x,t)$ is a system with an equilibrium point $x_e = 0$. Let $V(x,t) : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ be a nonnegative function, called a Lyapunov function, with derivative $\dot{V}$ along the trajectories of the system $\dot{x} = f(x,t)$.

If $V(x,t)$ is locally positive definite and there exists some $\varepsilon > 0$ such that $\dot{V}(x,t) \leq 0$ for all x such that $||x|| < \varepsilon$ and for all t, then $x_e = 0$ is a locally stable equilibrium point in the sense of Lyapunov.

This portion of the basic theorem of Lyapunov, also known as the direct method, allows us to conclude local stability using a locally positive definite *Lyapunov function* - we don't need to know the solution to the system! There are stronger versions of this theorem that allow us to conclude stronger forms of stability!

Problem 2: The dynamics of a simple pendulum with some friction are given by:

$$\ddot{\theta} = -a \sin\theta - \beta\dot{\theta} \tag{6}$$

A potential Lyapunov function for this system is based off of energy:

$$V(\theta, \dot{\theta}) = \frac{1}{2}\dot{\theta}^2 + a(1 - \cos\theta) \tag{7}$$

Show that this function is locally positive definite. ($V(\theta, \dot{\theta}) \geq 0$ for some region around $(\theta, \dot{\theta}) = (0, 0)$).

Solution: This function has two components: a quadratic component, which is always ≥ 0, and a trigonometric component. From our knowledge of trig, the component $a(1 - \cos\theta)$ is bounded between 0 and $2a$ for all values of θ. Thus, since both components are ≥ 0 for some region of $\theta, \dot{\theta}$, and the entire function is zero for $\theta = \dot{\theta} = 0$, we conclude V is LPDF.

Problem 3: Take the time derivative of $V(\theta, \dot{\theta})$ subject to the constraint that $\ddot{\theta} = -a\sin\theta - \beta\dot{\theta}$, and show that for certain values of $(\theta, \dot{\theta})$ near the $(\theta, \dot{\theta}) = (0, 0)$, $\dot{V} \leq 0$. What can you conclude about the stability of the equilibrium point?

Solution: Taking the derivative of V along the trajectories of the system:

$$\begin{aligned} \dot{V} &= \dot{\theta}\ddot{\theta} + a\dot{\theta}\sin\theta && (8) \\ &= \dot{\theta}(-a\sin\theta - \beta\dot{\theta}) + a\dot{\theta}\sin\theta && (9) \\ &= -\beta\dot{\theta}^2 && (10) \end{aligned}$$

This is less than or equal to zero for some region of $\theta, \dot{\theta}$. Thus, the equilibrium point is at least locally stable in the sense of Lyapunov.

5 Optimization

In this class, we will be working a great deal with optimization. Setting up an optimization problem correctly allows us to plug our system into a solver and receive appropriate outputs, usually control inputs for our robot.

Our optimization problem has 3 major parts:

1. Objective function: this is what the optimization problem seeks to minimize/maximize. It is often also called the cost function. For example, we may want to minimize our distance to some goal.
2. Constraints: Limitations we set to the system. For example, a constraint for an autonomous car might be an obstacle in the path that we must navigate around.
3. Decision variables: These will be the outputs of the optimizer. What do we want to decide upon? For example, these could be control inputs for our robot.

Optimization theory is a major branch of research. The mathematical structure of an optimization problem with decision variable (potentially vector) x takes the following form:

$$x^* = \arg\min_{x \in \mathbb{R}^n} f(x) \text{ (cost function)} \tag{11}$$

$$\text{s.t. } Ax \leq b \text{ and } Cx = d \text{ (optimization constraint)} \tag{12}$$