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The Kinematics of Contact and Grasp

Abstract

The kinematics of contact describe the motion of a point of contact over the surfaces of two contacting objects in response to a relative motion of these objects. Using concepts from differential geometry, I derive a set of equations, called the contact equations, that embody this relationship. I employ the contact equations to design the following applications to be executed by an end-effector with tactile sensing capability: (1) determining the curvature form of an unknown object at a point of contact; and (2) following the surface of an unknown object. The contact equations also serve as a basis for an investigation of the kinematics of grasp. I derive the relationship between the relative motion of two fingers grasping an object and the motion of the points of contact over the object surface. Based on this analysis, we explore the following applications: (1) rolling a sphere between two arbitrarily shaped fingers; (2) fine grip adjustment (i.e., having two fingers that grasp an unknown object locally optimize their grip for maximum stability).

1. Introduction

A kinematic relation describes the dependence of one set of motion parameters on another such set due to the geometry and mechanics of the physical world. One prominent example of a kinematic relation is that of the kinematic chain, which is discussed in most texts on robotics, including Craig (1986). A kinematic chain is a coordinate transformation that relates the position and orientation of an end-effector to the joint angles and displacements of the attached manipulator.

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Another example of a kinematic relation is the grip Jacobian defined in Salisbury (1982). This linear transformation calculates the velocity of an object in the grasp of the fingers of a hand given the velocities of the joints of the fingers.

In this paper I discuss the kinematics of rigid bodies that maintain contact while in relative motion. In particular, I examine the kinematic relation between the relative motion of two objects and the motion of a point of contact over the surfaces of these objects. Investigations of this kinematic relation have previously put simplifying restrictions on the shapes of the objects (e.g., flat, spherical, or two-dimensional) and/or the type of relative motion (pure sliding or pure rolling) (e.g., see Cai and Roth 1986; Kerr and Roth 1986; Bajcsy 1984; Mason 1981). A general description of this kinematic relation has been derived by myself (Montana 1986) and, independently, by Cai and Roth (1987). Using methods from differential geometry, I provide a formulation and solution of the kinematics of contact that is more mathematically rigorous and concise.

The contact equations are the equations that I derive which encapsulate this kinematic relation. Based on the contact equations, I investigate two tasks for a single end-effector with tactile sensing capability. (Tactile sensing is needed because it allows us to measure the position of a point of contact on the end-effector surface [Fearing and Hollerbach 1985].) First, I describe how such an end-effector can determine the curvature form of an unknown object at a point of contact by performing rotational probes and measuring the motion of the point of contact across its own surface. The curvature form of the object is estimated as that which fits these measurements in a least-squares way. Second, I show how to have such an end-effector follow the surface of an unknown object. Tactile data is used to close a loop around the kinematics of contact and steer the point of contact as desired on the end-effector surface. This contour-following algorithm adapts to the unknown and changing curvature of the object. A contour-following scheme based on the kinematics of contact is also presented in Cai and Roth (1987). However, there it is assumed that the curvature of the object is already known, and they are therefore solving a different (and easier) problem.

I also use the contact equations to investigate the kinematics of grasp. This is the problem of manipulating an object with a number of independent end-effectors, usually the fingers of a hand. Most research on mechanical hands has focused on particular hands and/or particular applications (Hanafusa and Asada 1977; Okada 1982). A general theory of manipulation was formulated in Salisbury (1982). Assuming stationary points of contact, Salisbury's grip Jacobian determines the finger joint velocities needed to produce a given velocity of the grasped object relative to the palm. In Kerr and Roth (1986), Salisbury's analysis is extended to allow rolling contact. However, the kinematic relation of interest is still the same. Allowing the points of contact to move just provides extra freedom in how to choose the joint motions to produce a desired object motion. Like Kerr and Roth, I examine grasps with rolling contact, but I derive the kinematic relation between the relative motion of two fingers grasping an object and the motion of the points of contact on the object surface. To do this, I apply the contact equations at each point of contact and perform suitable coordinate transformations to combine the two sets of equations into one.

I use this kinematic relation to investigate a couple of tasks for two fingers. First, I examine the problem of rolling a spherical object between two arbitrarily shaped fingers. This problem reduces to choosing a relative motion of the fingers such that the two points of contact remain diametrically opposed on the object surface. I also investigate the task of fine grip adjustment, showing how two fingers grasping an unknown object can locally optimize their respective points of contact with the object to achieve maximum stability. This is done by iterating on the following two steps: (1) determine the local geometry (position, surface orientation, and curvature) of the object at each point of contact, and (2) move the points of contact to new positions on the object surface so as to improve a certain grip stability criterion.

2. Mathematical Background

In this section I discuss concepts concerning rigid-body motion (Craig 1986) and the geometry of curves and surfaces (Spivak 1979).

NOTATION 1 Let C_{s_1} and C_{s_2} be two coordinate frames, where s_1 and s_2 are arbitrary subscripts. Then, $\mathbf{p}_{s_2s_1}$ and $R_{s_2s_1}$ denote the position and orientation of C_{s_1} relative to C_{s_2}. Furthermore, $\mathbf{v}_{s_2s_1} = R^T_{s_2s_1}$ and $\Omega_{s_2s_1} = R^T_{s_2s_1}\dot{R}_{s_2s_1}$ are the translational velocity and rotational velocity of C_{s_1} relative to C_{s_2}. The vector form of angular velocity is denoted by $\omega_{s_2s_1}$. For instance, $\mathbf{p}_{21}$, R_{21}, $\mathbf{v}_{21}$, and Ω_{21} describe the motion of a frame named C_1 relative to a frame named C_2. Similarly, $\mathbf{p}_{a_1b}$, R_{a_1b}, $\mathbf{v}_{a_1b}$, and Ω_{a_1b} are the motion parameters of C_b relative to C_{a_1}.

PROPOSITION 1 *Consider three coordinate frames C_1, C_2, and C_3. The following relation exists between their relative velocities:*

$$\begin{aligned} \mathbf{v}_{13} &= R^T_{23}\mathbf{v}_{12} + R^T_{23}\Omega_{12}\mathbf{p}_{23} + \mathbf{v}_{23}, \\ \Omega_{13} &= R^T_{23}\Omega_{12}R_{23} + \Omega_{23}. \end{aligned} \tag{1}$$

Equivalently, in terms of the vector form of angular velocity, we have

$$\begin{aligned} \mathbf{v}_{13} &= R^T_{23}(\mathbf{v}_{12} + \omega_{12} \times \mathbf{p}_{23}) + \mathbf{v}_{23}, \\ \omega_{13} &= R^T_{23}\omega_{12} + \omega_{23}. \end{aligned} \tag{2}$$

Proof: The positions and orientations are composed according to

$$\mathbf{p}_{13} = \mathbf{p}_{12} + R_{12}\mathbf{p}_{23}, \qquad R_{13} = R_{12}R_{23}. \tag{3}$$

Hence, the translational and rotational velocities can be expressed as

$$\begin{aligned} \mathbf{v}_{13} &= R^T_{13}\dot{\mathbf{p}}_{13} = R^T_{23}R^T_{12}(\dot{\mathbf{p}}_{12} + \dot{R}_{12}\mathbf{p}_{23} + R_{12}\dot{\mathbf{p}}_{23}) \\ &= R^T_{23}\mathbf{v}_{12} + R^T_{23}\Omega_{12}\mathbf{p}_{23} + \mathbf{v}_{23}, \end{aligned} \tag{4}$$

$$\begin{aligned} \Omega_{13} &= R^T_{13}\dot{R}_{13} = R^T_{23}R^T_{12}(\dot{R}_{12}R_{23} + R_{12}\dot{R}_{23}) \\ &= R^T_{23}\Omega_{12}R_{23} + \Omega_{23}. \end{aligned} \tag{5}$$

Definition 1.

A *coordinate patch* S_0 for a surface $S \subset \Re^3$ is an open, connected subset of S with the following property: There exists an open subset U of $\Re^2$ and an invertible map $f: U \rightarrow S_0 \subset \Re^3$ such that the partial derivatives $f_u(\mathbf{u})$ and $f_v(\mathbf{u})$ are linearly independent for all $\mathbf{u} = (u, v) \in U$. The pair (f, U) is called a *coordinate system* for S_0. The *coordinates* of a point $s \in S_0$ are $(u, v) = f^{-1}(s)$. A *2-manifold embedded in* $\Re^3$ (which we henceforth call a *manifold*) is a surface $S \subset \Re^3$ that can be written $S = \cup_{i=1}^{n} S_i$, where the S_i's are coordinate patches for S. The set $\{S_i\}_{i=1}^{n}$ is called an *atlas* for S.

Definition 2.

A *Gauss map* (or *normal map*) for a manifold S is a continuous map $g: S \rightarrow S^2 \subset \Re^3$ such that for every $s \in S$, $g(s)$ is perpendicular to S at s. (Recall that S^2 is the unit sphere.) An *orientable* manifold S is one for which a Gauss map exists. When S is the surface of a solid object, we call the Gauss map that points outward the *outward normal map* and the one that points inward the *inward normal map*.

Definition 3.

Consider a manifold S with Gauss map g, a coordinate patch S_0 for S, and a coordinate system (f, U) for S_0. The coordinate system (f, U) is *orthogonal* if $f_u(\mathbf{u}) \cdot f_v(\mathbf{u}) = 0$ for all $\mathbf{u} \in U$. When (f, U) is orthogonal, we can define the *normalized Gauss frame* at a point $\mathbf{u} \in U$ as the coordinate frame with origin at $f(\mathbf{u})$ and coordinate axes

$$\mathbf{x}(\mathbf{u}) = f_u(\mathbf{u})/\|f_u(\mathbf{u})\|, \quad \mathbf{y}(\mathbf{u}) = f_v(\mathbf{u})/\|f_v(\mathbf{u})\|, \quad \mathbf{z}(\mathbf{u}) = g(f(\mathbf{u})). \tag{6}$$

Note that the coordinate axes are functions mapping U to $\Re^3$. We call an orthogonal coordinate system (f, U) *right-handed* if its induced normalized Gauss frame is everywhere right-handed.

NOTE 1 1. For any coordinate patch with an associated Gauss map there exists a right-handed, orthogonal coordinate system.

2. The normalized Gauss frame is an example of what Cartan called a moving frame (Cartan 1946). Cartan used moving frames to define the curvature form and torsion form, and we now adapt his definitions into the present context.

Definition 4.

Consider a manifold S with Gauss map g, coordinate patch S_0, and orthogonal coordinate system (f, U). At a point $s \in S_0$, the *curvature form* K is defined as the 2×2 matrix

$$K = [\mathbf{x}(\mathbf{u}), \mathbf{y}(\mathbf{u})]^T[\mathbf{z}_u(\mathbf{u})/\|f_u(\mathbf{u})\|, \mathbf{z}_v(\mathbf{u})/\|f_v(\mathbf{u})\|], \tag{7}$$

where $\mathbf{u} = f^{-1}(s)$. The *torsion form* T at s is the 1×2 matrix

$$T = \mathbf{y}(\mathbf{u})^T[\mathbf{x}_u(\mathbf{u})/\|f_u(\mathbf{u})\|, \mathbf{x}_v(\mathbf{u})/\|f_v(\mathbf{u})\|]. \tag{8}$$

We define the *metric* M at s as the 2×2 diagonal matrix

$$M = \mathrm{diag}(\|f_u(\mathbf{u})\|, \|f_v(\mathbf{u})\|). \tag{9}$$

Our metric is the square root of the Riemannian metric (Spivak 1979).

EXAMPLE 1 Consider the set

$$U = \{(u, v) | -\pi/2 < u < \pi/2, -\pi < v < \pi\} \tag{10}$$

and the map

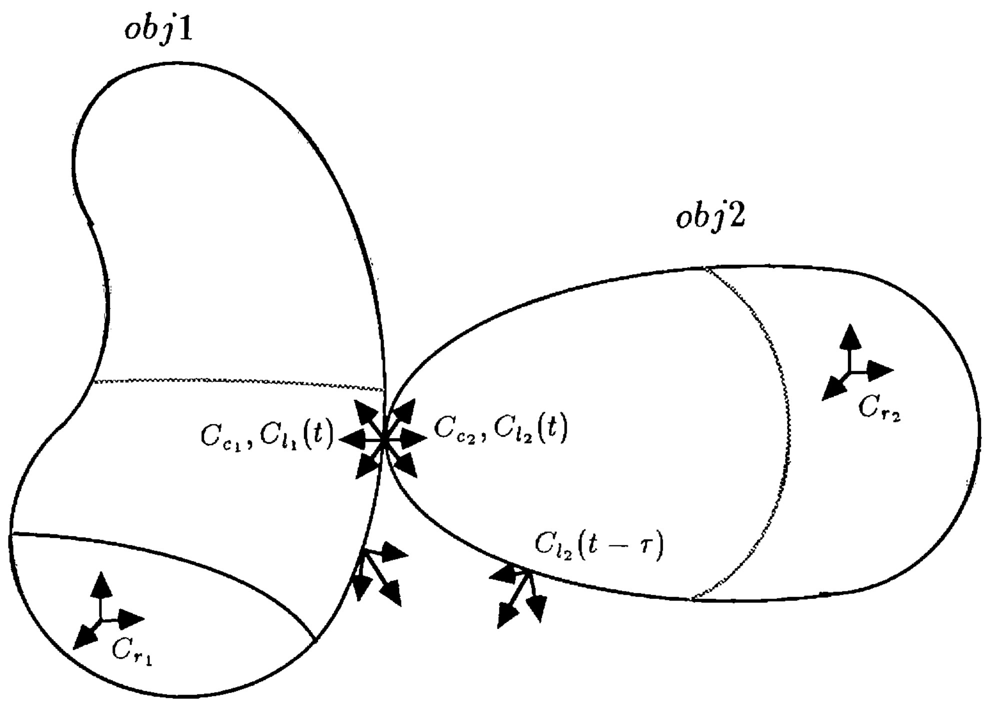
$$f: U \rightarrow \Re^3, \quad (u, v) \mapsto (R \cos u \cos v, -R \cos u \sin v, R \sin u) \tag{11}$$

for some $R > 0$. Let $S_0 = f(U)$. The reader can verify that (f, U) is a coordinate system for S_0. Let S be the sphere of radius R. Then S_0 is a coordinate patch for S. The coordinates u and v are known as the latitude and longitude, respectively. We can define another map

$$\tilde{f}: U \rightarrow \Re^3, \quad (u, v) \mapsto (-R \cos u \cos v, R \sin u, R \cos u \sin u). \tag{12}$$

Let $\tilde{S}_0 = \tilde{f}(U)$. Then $\{S_0, \tilde{S}_0\}$ is an atlas for S. Hence,

Fig. 1. The coordinate frames at time t *(with* $\tau > 0$*).*



S is a manifold. If we view the sphere as the surface of a ball, then the outward normal map is

$$g: S \to S^2, \qquad \mathbf{v} \mapsto (1/R)\mathbf{v}. \tag{13}$$

With this normal map, (f, U) is right-handed. It can be shown that (f, U) is an orthogonal coordinate system. Therefore, the normalized Gauss frame exists for all $\mathbf{u} = (u, v) \in U$. Its coordinate vectors are

$$\mathbf{x}(\mathbf{u}) = \begin{bmatrix} -\sin u \cos v \\ \sin u \sin v \\ \cos u \end{bmatrix}, \quad \mathbf{y}(\mathbf{u}) = \begin{bmatrix} -\sin v \\ -\cos v \\ 0 \end{bmatrix}, \quad \mathbf{z}(\mathbf{u}) = \begin{bmatrix} \cos u \cos v \\ -\cos u \sin v \\ \sin u \end{bmatrix}. \tag{14}$$

On the spherical surface of the earth, the x-, y-, and z-directions are called north, west, and up, respectively. The curvature form, torsion form, and metric are

$$K = \begin{bmatrix} 1/R & 0 \\ 0 & 1/R \end{bmatrix}, \quad T = \begin{bmatrix} 0 & \frac{-\tan u}{R} \end{bmatrix}, \quad M = \begin{bmatrix} R & 0 \\ 0 & R\cos u \end{bmatrix}. \tag{15}$$

3. The Kinematics of Contact

We now consider two rigid objects that move while maintaining contact with each other. Rigid bodies will generally make contact at isolated points rather than over areas of their surfaces. In this section we investigate the motion of one of these points of contact across the surfaces of the objects in response to a relative motion of the objects.

Call the objects obj 1 and obj 2. Choose reference frames C_{r_1} and C_{r_2} fixed relative to obj 1 and obj 2, respectively. Let $S_1 \subset \Re^3$ and $S_2 \subset \Re^3$ be the embeddings of the surfaces of obj 1 and obj 2 relative to C_{r_1} and C_{r_2}, respectively. Surfaces S_1 and S_2 are orientable manifolds. Let g_1 and g_2 be the outward normal maps for S_1 and S_2. Choose atlases $\{S_{1_i}\}_{i=1}^{n_1}$ and $\{S_{2_j}\}_{j=1}^{n_2}$ for S_1 and S_2. Let (f_{1_i}, U_{1_i}) be an orthogonal, right-handed coordinate system for S_{1_i} with normal map g_1. Similarly, let (f_{2_j}, U_{2_j}) be an orthogonal, right-handed coordinate system for S_{2_j} with g_2.

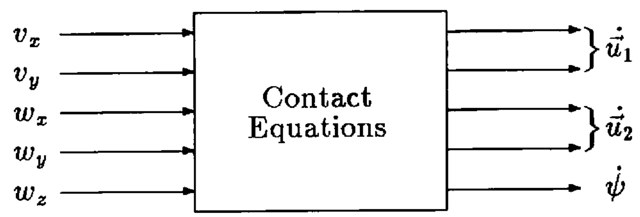
Let $c_1(t) \in S_1$ and $c_2(t) \in S_2$ be the positions at time t of the point of contact relative to C_{r_1} and C_{r_2}, respectively. In general, $c_1(t)$ will not remain in a single coordinate patch of the atlas $\{S_{1_i}\}_{i=1}^{n_1}$ for all time, and likewise for $c_2(t)$ and the atlas $\{S_{2_j}\}_{j=1}^{n_2}$. Therefore, we restrict our attention to an interval I such that $c_1(t) \in S_{1_i}$ and $c_2(t) \in S_{2_j}$ for all $t \in I$ and some i and j. The coordinate systems (f_{1_i}, U_{1_i}) and (f_{2_j}, U_{2_j}) induce a normalized Gauss frame at all points in S_{1_i} and S_{2_j}. We define the contact frames, C_{c_1} and C_{c_2} as the coordinate frames that coincide with the normalized Gauss frames at $c_1(t)$ and $c_2(t)$, respectively, for all $t \in I$. We also define a continuous family of coordinate frames, two for each $t \in I$, as follows. Let the local frames at time t, $C_{l_1}(t)$ and $C_{l_2}(t)$, be the coordinate frames fixed relative to C_{r_1} and C_{r_2}, respectively, that coincide at time t with the normalized Gauss frames at $c_1(t)$ and $c_2(t)$ (see Fig. 1).

We now define the parameters that describe the 5 degrees of freedom for the motion of the point of contact. The coordinates of the point of contact relative to the coordinate systems (f_{1_i}, U_{1_i}) and (f_{2_j}, U_{2_j}) are given by $\mathbf{u}_1(t) = f_{1_i}^{-1}(c_1(t)) \in U_{1_i}$ and $\mathbf{u}_2(t) = f_{2_j}^{-1}(c_2(t)) \in U_{2_j}$. These account for 4 degrees of freedom. The final parameter is the angle of contact $\psi(t)$, which is defined as the angle between the x-axes of C_{c_1} and C_{c_2}. We choose the sign of ψ so that a rotation of C_{c_1} through angle $-\psi$ around its z-axis aligns the x-axes.

We describe the motion of obj 1 relative to obj 2 at time t, using the local coordinate frames $C_{l_1}(t)$ and $C_{l_2}(t)$. Let v_x, v_y, and v_z be the components of translational velocity of $C_{l_1}(t)$ relative to $C_{l_2}(t)$ at time t. Similarly, let ω_x, ω_y, and ω_z be the components of rotational velocity. Then v_x, v_y, v_z, ω_x, ω_y, and ω_z provide the 6 degrees of freedom for the relative motion between the objects (see Fig. 2).

The symbols K_1, T_1, and M_1 represent, respectively, the curvature form, torsion form, and metric at time t at the point $c_1(t)$ relative to the coordinate system

Fig. 2. Sliding contact.



(f_{1_i}, U_{1_i}). We can analogously define K_2, T_2, and M_2. We also let

$$R_\psi = \begin{bmatrix} \cos\psi & -\sin\psi \\ -\sin\psi & -\cos\psi \end{bmatrix}, \qquad \tilde{K}_2 = R_\psi K_2 R_\psi. \quad (16)$$

Note that R_ψ is the orientation of the x- and y-axes of C_{c_1} relative to the x- and y-axes of C_{c_2}. Hence, $\tilde{K}_2$ is the curvature of obj 2 at the point of contact relative to the x- and y-axes of C_{c_1}. Call $K_1 + \tilde{K}_2$ the *relative curvature form.*

THEOREM 1 *At a point of contact, if the relative curvature form is invertible, then the point of contact and angle of contact evolve according to*

$$\dot{\mathbf{u}}_1 = M_1^{-1}(K_1 + \tilde{K}_2)^{-1}\left(\begin{bmatrix} -\omega_y \\ \omega_x \end{bmatrix} - \tilde{K}_2 \begin{bmatrix} v_x \\ v_y \end{bmatrix}\right), \quad (17)$$

$$\dot{\mathbf{u}}_2 = M_2^{-1}R_\psi(K_1 + \tilde{K}_2)^{-1}\left(\begin{bmatrix} -\omega_y \\ \omega_x \end{bmatrix} + K_1 \begin{bmatrix} v_x \\ v_y \end{bmatrix}\right), \quad (18)$$

$$\dot{\psi} = \omega_z + T_1 M_1 \dot{\mathbf{u}}_1 + T_2 M_2 \dot{\mathbf{u}}_2, \quad (19)$$

$$0 = v_z. \quad (20)$$

Proof: Recall the notation introduced in Notation 1. Since $C_{l_1}(t)$ is fixed relative to C_{r_1}, the velocity at time t of $C_{l_1}(t)$ relative to C_{r_1} is given by $\mathbf{v}_{r_1 l_1} = 0$ and $\Omega_{r_1 l_1} = 0$. Therefore, according to Proposition 1,

$$\mathbf{v}_{r_1 c_1} = \mathbf{v}_{l_1 c_1}, \qquad \Omega_{r_1 c_1} = \Omega_{l_1 c_1}. \quad (21)$$

Similarly, we find that

$$\mathbf{v}_{r_2 c_2} = \mathbf{v}_{l_2 c_2}, \qquad \Omega_{r_2 c_2} = \Omega_{l_2 c_2}. \quad (22)$$

At time t the position and orientation of C_{c_1} relative to

$C_{l_1}(t)$ are $\mathbf{r}_{l_1c_1} = 0$ and $R_{l_1c_1} = I$. Hence, Proposition 1 states that

$$\mathbf{v}_{l_2c_1} = \mathbf{v}_{l_1c_1} + \mathbf{v}_{l_2l_1}, \qquad \Omega_{l_2c_1} = \Omega_{l_1c_1} + \Omega_{l_2l_1}. \tag{23}$$

Since $\mathbf{p}_{c_2c_1} = 0$, according to Proposition 1,

$$\begin{aligned} \mathbf{v}_{l_2c_1} &= \mathbf{v}_{c_2c_1} + R^{\mathrm{T}}_{c_2c_1}\mathbf{v}_{l_2c_2}, \\ \Omega_{l_2c_1} &= \Omega_{c_2c_1} + R^{\mathrm{T}}_{c_2c_1}\Omega_{l_2c_2}R_{c_2c_1}. \end{aligned} \tag{24}$$

Combining Eqs. (21-24) yields

$$\mathbf{v}_{r_1c_1} + \mathbf{v}_{l_2l_1} = \mathbf{v}_{c_2c_1} + R^{\mathrm{T}}_{c_2c_1}\mathbf{v}_{r_2c_2}, \tag{25}$$

$$\Omega_{r_1c_1} + \Omega_{l_2l_1} = \Omega_{c_2c_1} + R^{\mathrm{T}}_{c_2c_1}\Omega_{r_2c_2}R_{c_2c_1}. \tag{26}$$

We now find the values of each of the quantities in Eqs. (25) and (26) in terms of the contact parameters and motion parameters. To start, we observe that

$$\begin{aligned} R_{c_2c_1} &= \begin{bmatrix} R_\psi & 0 \\ 0 & -1 \end{bmatrix}, \quad \mathbf{v}_{c_2c_1} = 0, \\ \Omega_{c_2c_1} &= \begin{bmatrix} 0 & -\dot{\psi} & 0 \\ \dot{\psi} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \end{aligned} \tag{27}$$

By the definition for v_x, v_y, v_z, ω_x, ω_y, and ω_z we gave above,

$$\mathbf{v}_{l_2l_1} = \begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix}, \qquad \Omega_{l_2l_1} = \begin{bmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{bmatrix}. \tag{28}$$

To examine the motion of C_{c_1} relative to C_{r_1}, let $\mathbf{x}_1(\mathbf{u}_1)$, $\mathbf{y}_1(\mathbf{u}_1)$, and $\mathbf{z}_1(\mathbf{u}_1)$ be the coordinate vectors of the normalized Gauss frame for obj 1 at the point $\mathbf{u}_1 \in U_{1i}$. Then,

$$\begin{aligned} \mathbf{p}_{r_1c_1} &= c_1(t) = f_{1_i}(\mathbf{u}_1(t)), \\ R_{r_1c_1} &= [\mathbf{x}_1(\mathbf{u}_1(t)), \mathbf{y}_1(\mathbf{u}_1(t)), \mathbf{z}_1(\mathbf{u}_1(t))], \end{aligned} \tag{29}$$

$$\begin{aligned} \mathbf{v}_{r_1c_1} &= R^{\mathrm{T}}_{r_1c_1}\dot{\mathbf{p}}_{r_1c_1} = [\mathbf{x}_1(\mathbf{u}_1), \mathbf{y}_1(\mathbf{u}_1), \mathbf{z}_1(\mathbf{u}_1)]^{\mathrm{T}} \\ &\quad \times [(f_{1_i})_u(\mathbf{u}_1), (f_{1_i})_v(\mathbf{u}_1)]\dot{\mathbf{u}}_1 = \begin{bmatrix} M\dot{\mathbf{u}}_1 \\ 0 \end{bmatrix}, \end{aligned} \tag{30}$$

$$\Omega_{c_1r_1} = R^{\mathrm{T}}_{c_1r_1}\dot{A}_{c_1r_1} \tag{31}$$

$$\begin{aligned} &= [\mathbf{x}_1, \mathbf{y}_1, \mathbf{z}_1]^{\mathrm{T}} \\ &\quad \times [[(\mathbf{x}_1)_u, (\mathbf{x}_1)_v]\dot{\mathbf{u}}_1, [(\mathbf{y}_1)_u, (\mathbf{y}_1)_v]\dot{\mathbf{u}}_1, [(\mathbf{z}_1)_u, (\mathbf{z}_1)_v]\dot{\mathbf{u}}_1] \end{aligned} \tag{32}$$

$$= \begin{bmatrix} 0 & -T_{1M_1}\dot{\mathbf{u}}_1 & K_1M_1\dot{\mathbf{u}}_1 \\ T_1M_1\dot{\mathbf{u}}_1 & 0 & \\ -(K_1M_1\dot{\mathbf{u}}_1)^{\mathrm{T}} & & 0 \end{bmatrix}. \tag{33}$$

We similarly find that

$$\begin{aligned} \mathbf{v}_{r_2c_2} &= \begin{bmatrix} M\dot{\mathbf{u}}_2 \\ 0 \end{bmatrix}, \\ \Omega_{r_2c_2} &= \begin{bmatrix} 0 & -T_2M_2\dot{\mathbf{u}}_2 & K_2M_2\dot{\mathbf{u}}_2 \\ T_2M_2\dot{\mathbf{u}}_2 & 0 & \\ -(K_2M_2\dot{\mathbf{u}}_2)^{\mathrm{T}} & & 0 \end{bmatrix}. \end{aligned} \tag{34}$$

Substituting Eqs. (27), (28), (30), (33), and (34) into Eqs. (25) and (26) and equating components, we get

$$M_1\dot{\mathbf{u}}_1 + \begin{bmatrix} v_x \\ v_y \end{bmatrix} = M_2\dot{\mathbf{u}}_2, \tag{35}$$

$$v_z = 0, \tag{36}$$

$$K_1M_1\dot{\mathbf{u}}_1 + \begin{bmatrix} \omega_y \\ -\omega_x \end{bmatrix} = -R_\psi K_2M_2\dot{\mathbf{u}}_2, \tag{37}$$

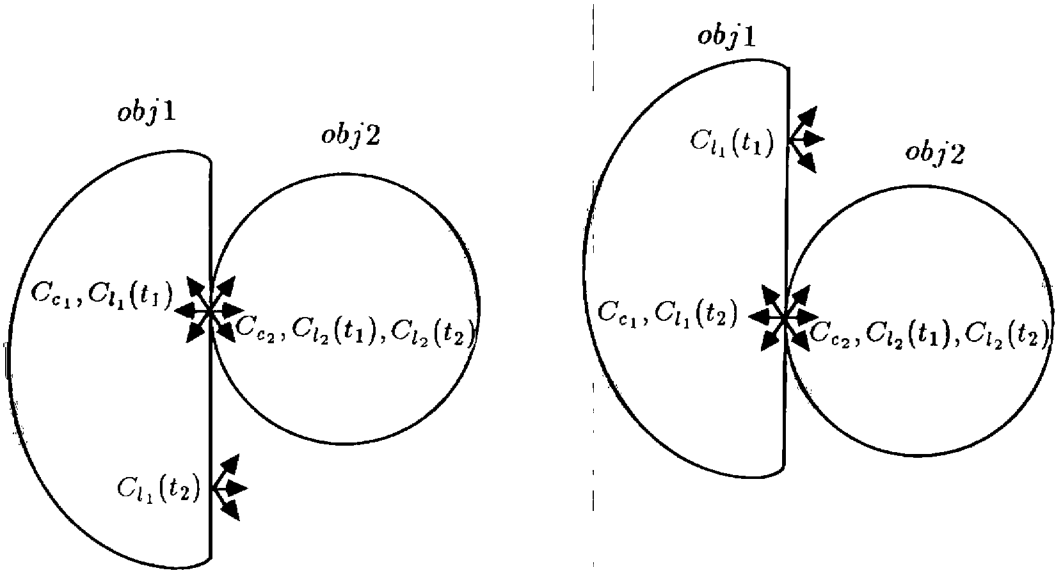
$$T_1M_1\dot{\mathbf{u}}_1 + \omega_z = \dot{\psi} - T_2M_2\dot{\mathbf{u}}_2. \tag{38}$$

After some algebraic manipulation, we can write Eqs. (35-38) in the form given in Eqs. (17-20).

We call Eqs. (17)-(19) the first, second, and third *contact equations* respectively. We call Eq. (20) the *kinematic constraint of contact* because it expresses the constraint on the relative motion necessary to maintain contact.

NOTE 2 For some of the applications discussed below, obj 2 will be an object of unknown shape. Hence, we will not be able to choose a coordinate system for it. We therefore now re-express the second contact equation in a form that is independent of the coordinate system chosen for obj 2. (The first contact equation is already in such a form.) Define $\tilde{s}_2 = R_\psi M_2\dot{\mathbf{u}}_2$. Then, $\tilde{s}_2$ is the rate at which the point of contact traverses arc length across the surface of obj 2 as measured relative to the x- and y-axes of the local

Fig. 3. Rolling without slipping.



(a) before: $t = t_1$ (b) after: $t = t_2$

coordinate frame of obj 1. This quantity is independent of the coordinate system chosen for obj 2. Substituting into the second contact equation gives

$$\dot{\mathbf{s}}_2 = (K_1 + \tilde{K}_2)^{-1}\left(\begin{bmatrix} -\omega_y \\ \omega_x \end{bmatrix} + K_1 \begin{bmatrix} v_x \\ v_y \end{bmatrix}\right). \quad (39)$$

EXAMPLE 2 Let obj 1 be an object whose surface has a planar coordinate patch. Choosing a Cartesian coordinate system for this coordinate patch yields $K_1 = 0$, $T_1 = 0$, and $M_1 = I$ at all points. Let obj 2 be a unit ball. Using the coordinate patch investigated in Example 1 gives values for the curvature form, torsion form, and metric of $K_2 = I$, $T_2 = [0, -\tan u]$, and $M_2 = \mathrm{diag}(1, \cos u)$. Let obj 1 and obj 2 be oriented so that at time t_0 the x-axis of $C_{l_1}(t)$ coincides with the x-axis of $C_{l_2}(t)$. Then at time t_0, $R_\psi = \mathrm{diag}(1, -1)$, and the contact equations are

$$\dot{\mathbf{u}}_1 = \begin{bmatrix} -\omega_y - v_x \\ \omega_x - v_y \end{bmatrix}, \quad \dot{\mathbf{u}}_2 = \begin{bmatrix} -\omega_y \\ -\omega_x \sec u_2 \end{bmatrix}, \quad (40)$$
$$\dot{\psi} = \omega_z - \omega_x \tan u_2,$$

where $\mathbf{u}_2 = [u_2, v_2]^{\mathrm{T}}$.

When there is sliding contact, $\omega_x = \omega_y = \omega_z = 0$. Therefore, Eq. (40) becomes

$$\dot{\mathbf{u}}_1 = \begin{bmatrix} -v_x \\ -v_y \end{bmatrix}, \quad \dot{\mathbf{u}}_2 = 0, \quad \dot{\psi} = 0. \quad (41)$$

This motion is pictured in Fig. 3.

When the relative motion is rolling without slipping, $v_x = v_y = \omega_z = 0$. Hence, Equation (40) is

$$\dot{\mathbf{u}}_1 = \begin{bmatrix} -\omega_y \\ \omega_x \end{bmatrix}, \quad \dot{\mathbf{u}}_2 = \begin{bmatrix} -\omega_y \\ -\omega_x \sec u_2 \end{bmatrix}, \quad (42)$$
$$\dot{\psi} = \omega_x \tan u_2.$$

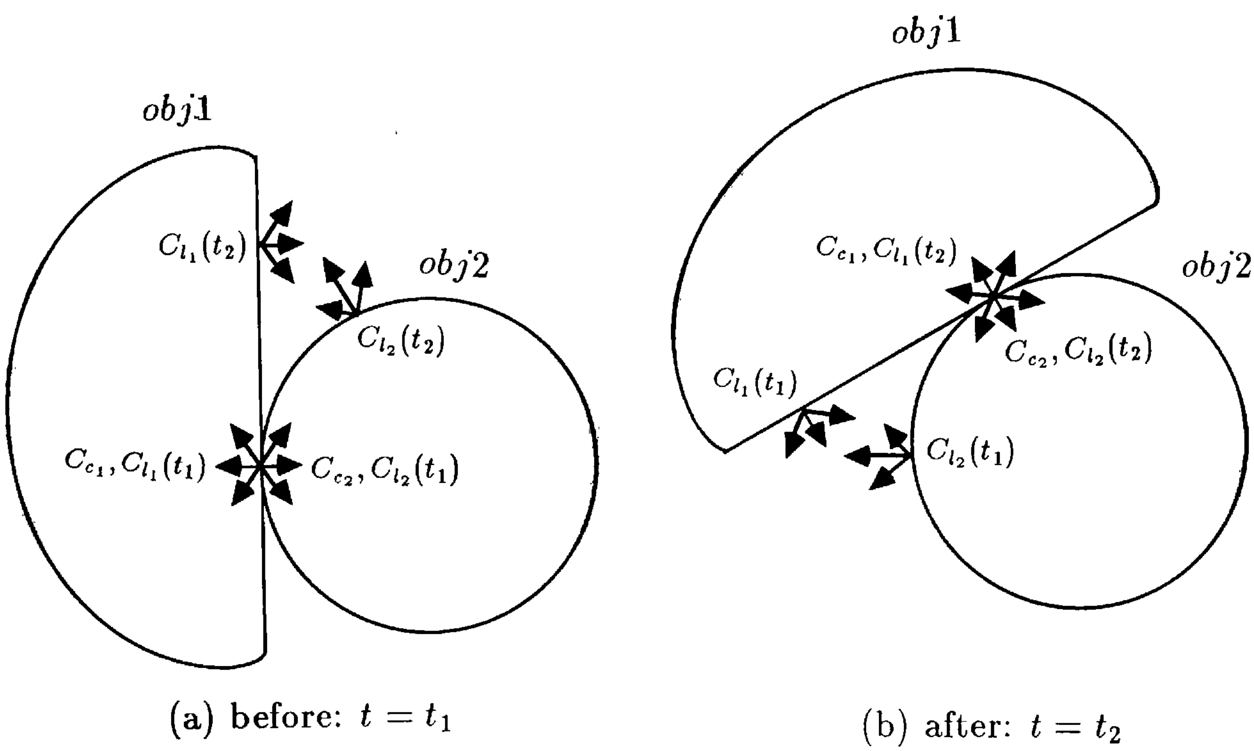
This motion is pictured in Fig. 4.

When the relative motion is rotation around the normal, $\omega_x = \omega_y = v_x = v_y = 0$. Then Eq. (40) becomes

$$\dot{\mathbf{u}}_1 = 0, \quad \dot{\mathbf{u}}_2 = 0, \quad \dot{\psi} = \omega_z. \quad (43)$$

For such motion the point of contact is fixed on both surfaces, and only the angle of contact changes.

Fig. 4. *The coordinate frames at time* t.



4. Application 1: Finding Curvature

Let obj 1 be a tactile sensor attached to a manipulator, and let obj 2 be an object of unknown shape. Assume that there is a single point of contact between them. We now discuss how to determine $\tilde{K}_2$, the curvature form of the unknown object at the point of contact, through a series of experiments. The ith experiment consists of rotating the sensor without slippage relative to the object through a small angle $[\Delta\theta_{x_i}, \Delta\theta_{y_i}, 0]^T$. Assume that the point of contact remains in one coordinate patch for the experiments. Then the tactile sensor can measure the resulting change in the coordinates of the point of contact on its surface $\Delta\mathbf{u}_{1_i}$. Since the inverse of the relative curvature form is symmetric, we can write it as

$$(K_1 + \tilde{K}_2)^{-1} = \begin{bmatrix} k_{r_1} & k_{k_2} \\ k_{r_2} & k_{r_3} \end{bmatrix}.$$

Because the shape of the sensor and the chosen coordinate system are known and the coordinates of the point of contact on the sensor surface can be measured, we can compute M_1 and K_1.

PROPOSITION 2 *Consider n such rotational probes. The values of* k_{r_1}, k_{r_2}, and k_{r_3} *that minimize the sum of the squares of the errors in the measurements of* $M_i\,\Delta\mathbf{u}_{1_i}$ *are given by*

$$\begin{bmatrix} k_{r_1} \\ k_{r_2} \\ k_{r_3} \end{bmatrix} = (A^T A)^{-1} A^T B, \tag{44}$$

where A and B are defined as

$$A = \begin{bmatrix} -\Delta\theta_{y_1} & \Delta\theta_{x_1} & 0 \\ 0 & -\Delta\theta_{y_1} & \Delta\theta_{x_1} \\ \cdots & \cdots & \cdots \\ -\Delta\theta_{y_n} & \Delta\theta_{x_n} & 0 \\ 0 & -\Delta\theta_{y_n} & \Delta\theta_{x_n} \end{bmatrix}, \quad B = \begin{bmatrix} M_1\Delta\mathbf{u}_{1_1} \\ \cdots \\ M_1\Delta\mathbf{u}_{1_n} \end{bmatrix}. \tag{45}$$

Proof: According to the first contact equation,

$$M_1 \Delta \mathbf{u}_{1_i} + e_i = (K_1 + \tilde{K}_2)^{-1} \begin{bmatrix} -\Delta\theta_{y_i} \\ \Delta\theta_{x_i} \end{bmatrix}, \tag{46}$$

where e_i is the error in the measurement of $M_1 \Delta \mathbf{u}_{1_i}$. This can be rewritten as

$$\begin{bmatrix} -\Delta\theta_{y_i} & \Delta\theta_{x_i} & 0 \\ 0 & -\Delta\theta_{y_i} & \Delta\theta_{x_i} \end{bmatrix} \begin{bmatrix} k_{r_1} \\ k_{r_2} \\ k_{r_3} \end{bmatrix} = M_1 \Delta \mathbf{u}_{1_i} + e_i. \tag{47}$$

Combining the results of all n experiments gives

$$A \begin{bmatrix} k_{r_1} \\ k_{r_2} \\ k_{r_3} \end{bmatrix} = B + \begin{bmatrix} e_1 \\ \cdots \\ e_n \end{bmatrix}. \tag{48}$$

The value of $[k_{r_1}, k_{r_2}, k_{r_3}]^T$ that minimizes the square of the error term is as shown in Eq. (44) (Campbell and Meyer 1979).

Given the inverse of the relative curvature form, we can solve for the curvature form of the unknown object as

$$\tilde{K}_2 = \begin{bmatrix} k_{r_1} & k_{r_2} \\ k_{r_2} & k_{r_3} \end{bmatrix}^{-1} - K_1. \tag{49}$$

5. Application 2: Contour Following

Take obj 1 to be an end-effector attached to a manipulator. Let obj 2 be some arbitrary object of unknown shape fixed relative to the base of the manipulator. We assume that the two objects meet at a single point of contact. We specify that the end-effector has tactile-sensing capability. With tactile sensing it is possible to measure the position of the point of contact on the surface of the end-effector. We also assume that we have proprioceptive sensors to measure the velocity of the end-effector relative to its base and hence relative to the fixed object.

In this section, we describe a closed-loop servosystem that drives the end-effector to steer the point of contact to some desired location on its own surface while following the surface of the unknown object. The main problem in designing such a servosystem is that the contact equations depend on the curvature form of the object whose shape is unknown. Our servosystem adapts to the changing shape of the unknown object and provides a partial estimate of its curvature form.

We start by choosing one coordinate patch on the surface of the end-effector in which we try to maintain the point of contact. (For human fingers, this coordinate patch would be the fingertip.) This allows us to always specify the position of the point of contact on the end-effector by its coordinates in this coordinate patch.

We assume that we can command the manipulator to produce any desired values for $\dot{v}_x$ $\dot{v}_y$, $\dot{w}_x$, $\dot{w}_y$, and $\dot{w}_z$. Let the velocity parameter C be an arbitrarily chosen two-vector. Choose the set point $\mathbf{u}_s$ to be a two-vector, which is the coordinates of some point in the selected coordinate patch for the end-effector. Define

$$e_1 = \begin{bmatrix} -\omega_y \\ \omega_x \end{bmatrix} + K_1 \begin{bmatrix} v_x \\ v_y \end{bmatrix} - C, \qquad e_2 = \mathbf{u}_1 - \mathbf{u}_s. \tag{50}$$

Let $(e_1)_m$, $(e_2)_m$, and $(\dot{e}_2)_m$ be the measured values of e_1, e_2, and $\dot{e}_2$, respectively.

Proposition 3 *If $\dot{K}_1 \approx 0$ and $\dot{C} \approx 0$ (i.e., K_1 and C are quasi-static), then the control law CL1,*

$$\begin{bmatrix} -\dot{w}_y \\ \dot{w}_x \end{bmatrix} + K_1 \begin{bmatrix} \dot{v}_x \\ \dot{v}_y \end{bmatrix} = -a_1(e_1)_m - a_2 \int (e_1)_m \, dt \tag{51}$$

with a_1 and a_2 positive constants, will steer e_1 to zero.

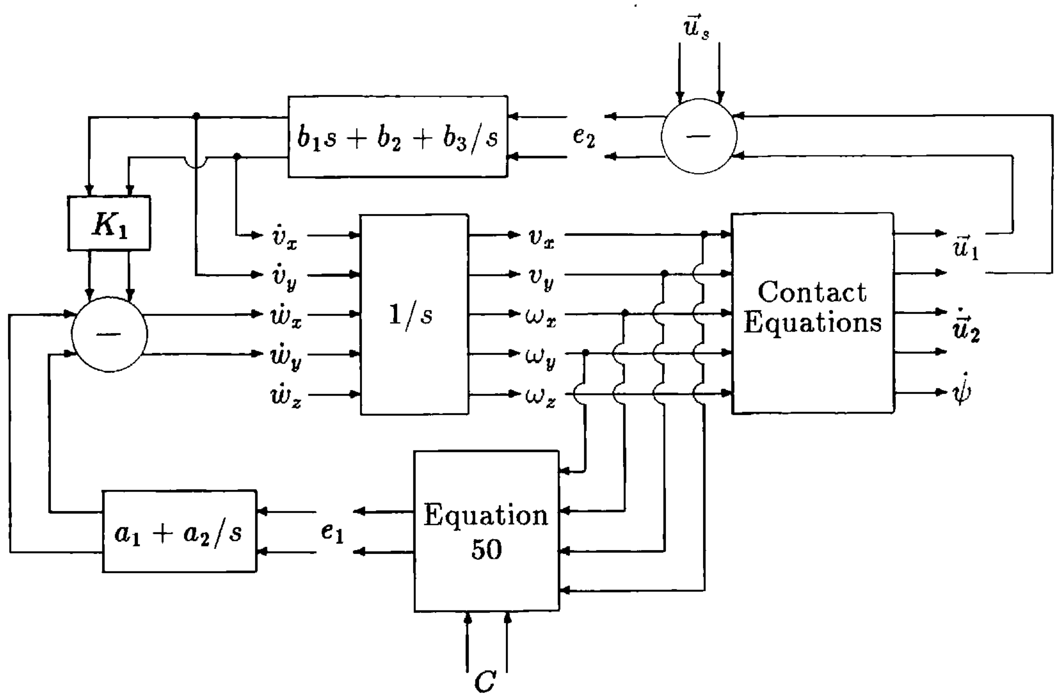
Proof: Differentiating the expression for e_1 in Eq. (50) gives

$$\begin{aligned} \dot{e}_1 &= \begin{bmatrix} -\dot{w}_y \\ \dot{w}_x \end{bmatrix} + K_1 \begin{bmatrix} \dot{v}_x \\ \dot{v}_y \end{bmatrix} \\ &= -a_1(e_1)_m - a_2 \int (e_1)_m \, dt. \end{aligned} \tag{52}$$

This is a proportional-integral (PI) system, which is known to steer e_1 to zero.

Proposition 4 *If M_1, K_1, $\tilde{K}_2$, $\mathbf{u}_s$, and e_1 are all quasi-static, then the control law CL2,*

Fig. 5. Closed-loop contour following.



$$\begin{bmatrix} \dot{v}_x \\ \dot{v}_y \end{bmatrix} = M_1 \left(b_1(\dot{e}_2)_m + b_2(e_2)_m + b_3 \int (e_2)_m \, dt \right) \quad (53)$$

with b_1, b_2, and b_3 positive constants, steers e_2 to zero.

Proof: The first contact equation can be written as

$$\dot{e}_2 + \dot{\mathbf{u}}_s = \dot{\mathbf{u}}_1 = M_1^{-1}(K_1 + \tilde{K}_2)^{-1} \times \left(C - (K_1 + \tilde{K}_2) \begin{bmatrix} v_x \\ v_y \end{bmatrix} + e_1 \right). \quad (54)$$

Differentiating Eq. (54) gives

$$\ddot{e}_2 = -M_1^{-1} \begin{bmatrix} \dot{v}_x \\ \dot{v}_y \end{bmatrix} = b_1(\dot{e}_2)_m + b_2(e_2)_m + b_3 \int (e_2)_m \, dt. \quad (55)$$

This is a proportional-integral-derivative (PID) system, which is known to steer e_2 to zero.

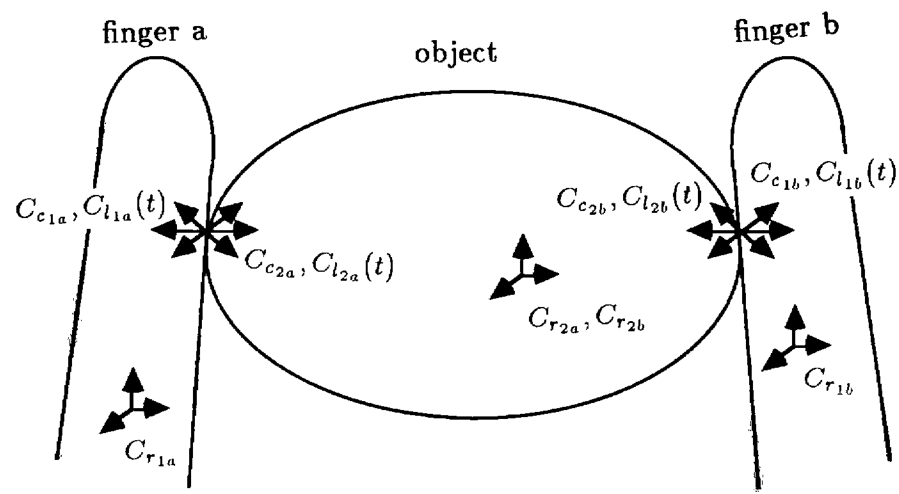
Combining Propositions 3 and 4 gives the following theorem.

THEOREM 2 *Assume that M_1, K_1, $\tilde{K}_2$, C, and $\mathbf{u}_s$ are all quasi-static and that the time scale for control law CL1 is small enough compared to that for CL2 so that CL1 appears to always be in steady state from the viewpoint of CL2. Then the control law obtained by combining CL1 and CL2 steers e_1 and e_2* to zero.

The quasi-static assumptions need *not* hold at all times. Any deviation from these assumptions causes a disturbance on the system that, if not too large, is compensated by the closed-loop control.

The control scheme of Theorem 2 is pictured in Fig. 5. The time scale of the lower loop is smaller than that of the upper loop. The free parameters in this system are C, $\mathbf{u}_s$, and $\dot{\omega}_z$. This contour-following algorithm is discussed further in Montana (1986). There it is shown how we can vary these free parameters in order to have the point of contact follow a line of curvature on the object surface. Also described in Montana (1986) is an initial implementation of this control scheme.

Fig. 6. Manipulation without slippage as an input-output system.



6. The Kinematics of Grasp

In this section we examine the problem of manipulating a rigid object with two end-effectors, which we refer to as fingers. We assume that the object has exactly one point of contact with each finger. We require that the fingers constantly grasp the object so as not to risk dropping it. Therefore, at each point of contact, the finger is constrained to roll without slipping so that static friction can be maintained.

We take the finger to be obj 1 and the object to be obj 2 at both points of contact. We refer to the two fingers as finger a and finger b. All symbols with subscript a refer to the point of contact between the object and finger a, and similarly for subscript b. The various coordinate frames are pictured in Fig. 6. The constraint that the fingers must roll without slipping can thus be expressed as $v_{xa} = v_{ya} = \omega_{za} = 0$ and $v_{xb} = v_{yb} = \omega_{zb} = 0$. To avoid the long subscripts induced by Notation 1, we let $\mathbf{p}_f$, R_f, $\mathbf{v}_f$, and $\mathbf{w}_f$ be the motion parameters of $C_{l_{1a}}(t)$ relative to $C_{l_{1b}}(t)$ at time t. Then $\mathbf{p}_f$, R_f, v_f, and $\mathbf{w}_f$ describe the relative motion of the two fingers at time t.

DEFINITION 5 We say that the two points of contact form a *grip* if

$$\begin{aligned} \cos^{-1}([0, 0, 1]\mathbf{p}_f/\|\mathbf{p}_f\|) &< \tan^{-1}(\kappa_s), \\ \cos^{-1}(-[0, 0, 1]R_f^T\mathbf{p}_f/\|\mathbf{p}_f\|) &< \tan^{-1}(\kappa_s) \end{aligned} \tag{56}$$

where κ_s is the static coefficient of friction (Mason 1982). (When the points of contact form a grip, the fingers can exert opposing forces and thus grasp the object.)

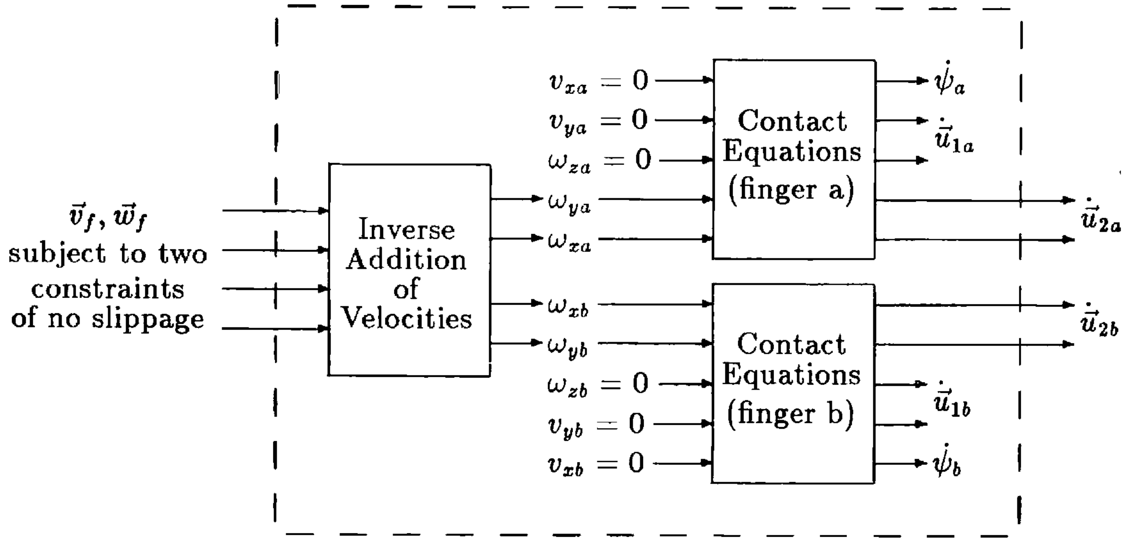
DEFINITION 6 We define the *addition of velocities* map $V(\mathbf{p}_f, R_f)$ as

$$V(\mathbf{p}_f, R_f): \Re^4 \to \Re^6, \quad \begin{bmatrix} \omega_{xa} \\ \omega_{ya} \\ \omega_{xb} \\ \omega_{yb} \end{bmatrix} \mapsto \begin{bmatrix} -R_f^T\left(\begin{bmatrix} \omega_{xb} \\ \omega_{yb} \\ 0 \end{bmatrix} \times \mathbf{p}_f\right) \\ -R_f^T \begin{bmatrix} \omega_{xb} \\ \omega_{yb} \\ 0 \end{bmatrix} + \begin{bmatrix} \omega_{xa} \\ \omega_{ya} \\ 0 \end{bmatrix} \end{bmatrix}. \tag{57}$$

THEOREM 3 *If the position and orientation of finger a relative to finger b are $\mathbf{p}_f$ and R_f and finger a and finger b roll without slipping relative to the object with angular velocity components ω_{xa}, ω_{ya}, ω_{xb}, and ω_{yb}, then the velocity of finger a relative to finger b is*

$$\begin{bmatrix} \mathbf{v}_f \\ \mathbf{w}_f \end{bmatrix} = V(\mathbf{p}_f, R_f)([\omega_{xa}, \omega_{ya}, \omega_{xb}, \omega_{yb}]^T). \tag{58}$$

Fig. 7. The contact equations as an input-output system.



Furthermore, if the points of contact form a grip, then $V(\mathbf{p}_f, R_f)$ is an injective map.

Proof: From Proposition 1 we find that

$$\mathbf{v}_{l_{2b}l_{1a}} = R^{\mathrm{T}}_{l_{1b}l_{1a}}(\mathbf{v}_{l_{2b}l_{1b}} + \mathbf{w}_{l_{2b}l_{1b}} \times \mathbf{p}_{l_{1b}l_{1a}}) + \mathbf{v}_{l_{1b}l_{1a}}, \quad (59)$$

$$\mathbf{w}_{l_{2b}l_{1a}} = R^{\mathrm{T}}_{l_{1b}l_{1a}}\mathbf{w}_{l_{2b}l_{1b}} + \mathbf{w}_{l_{1b}l_{1a}}, \quad (60)$$

Since the object is a rigid body, $\mathbf{v}_{l_{2b}l_{2a}} = \mathbf{w}_{l_{2b}l_{2a}} = 0$. Hence, Proposition 1 states that

$$\mathbf{v}_{l_{2b}l_{1a}} = \mathbf{v}_{l_{2a}l_{1a}}, \qquad \mathbf{w}_{l_{2b}l_{1a}} = \mathbf{w}_{l_{2a}l_{1a}}. \quad (61)$$

According to the statement of the theorem,

$$\begin{aligned} &\mathbf{w}_{l_{2a}l_{1a}} = [\omega_{xa}, \omega_{ya}, 0]^{\mathrm{T}}, \quad \mathbf{w}_{l_{2b}l_{1b}} = [\omega_{xb}, \omega_{yb}, 0]^{\mathrm{T}}, \\ &\mathbf{v}_{l_{2a}l_{1a}} = \mathbf{v}_{l_{2b}l_{1b}} = 0. \end{aligned} \quad (62)$$

Equations (59)–(62) can be combined to yield

$$\begin{aligned} 0 &= R_f^{\mathrm{T}}([\omega_{xb}, \omega_{yb}, 0]^{\mathrm{T}} \times \mathbf{p}_f) + \mathbf{v}_f, \\ [\omega_{xa}, \omega_{ya}, 0]^{\mathrm{T}} &= R_f^{\mathrm{T}}[\omega_{xb}, \omega_{yb}, 0]^{\mathrm{T}} + \mathbf{w}_f, \end{aligned} \quad (63)$$

which is equivalent to Eq. (58).

As to the injectivity of the addition of velocities map, let $\mathbf{p}_f = [p_{f_x}, p_{f_y}, p_{f_z}]^{\mathrm{T}}$. Then,

$$\begin{aligned} \cos^{-1}(p_{f_z}/\|\mathbf{p}_f\|) &= \cos^{-1}([0, 0, 1]\mathbf{p}_f/\|\mathbf{p}_f\|) \\ &< \tan^{-1}(\kappa_s) < \pi/2. \end{aligned} \quad (64)$$

We thus deduce that $p_{f_z} > 0$. From Eq. (63) we find that

$$-R_f\mathbf{v}_f = \begin{bmatrix} \omega_{xb} \\ \omega_{yb} \\ 0 \end{bmatrix} \times \begin{bmatrix} p_{f_x} \\ p_{f_y} \\ p_{f_z} \end{bmatrix} = \begin{bmatrix} \omega_{yb}p_{f_z} \\ -\omega_{xb}p_{f_z} \\ \omega_{xb}p_{f_y} - \omega_{yb}p_{f_x} \end{bmatrix}. \quad (65)$$

Therefore,

$$\mathbf{v}_f = 0 \Rightarrow \omega_{xb} = \omega_{yb} = 0. \quad (66)$$

From Eq. (63) we observe that

$$(\mathbf{w}_f = 0 \wedge \omega_{xb} = \omega_{yb} = 0) \Rightarrow \omega_{xa} = \omega_{ya} = 0. \quad (67)$$

Combining the logical implications of Eqs. (66) and (67) gives

$$\mathbf{v}_f = \mathbf{w}_f = 0 \Rightarrow \omega_{xb} = \omega_{yb} = \omega_{xa} = \omega_{ya} = 0. \quad (68)$$

So, the kernel of $V(\mathbf{p}_f, R_f)$ is zero-dimensional.

Since $V(\mathbf{p}_f, R_f)$ is injective, $V(\mathbf{p}_f, R_f)(\Re^4)$ is four-dimensional. Therefore, there exist two independent six-vectors $\mathbf{a}_1$ and $\mathbf{a}_2$ such that

$$\begin{bmatrix} \mathbf{v}_f \\ \mathbf{w}_f \end{bmatrix} \in V(\mathbf{p}_f, R_f)(\Re^4)$$

if and only if

$$\mathbf{a}_1 \cdot \begin{bmatrix} \mathbf{v}_f \\ \mathbf{w}_f \end{bmatrix} = 0, \qquad \mathbf{a}_2 \cdot \begin{bmatrix} \mathbf{v}_f \\ \mathbf{w}_f \end{bmatrix} = 0. \tag{69}$$

We call these conditions on the relative velocity of the sensors the *kinematic constraints of no slippage.* We can further conclude that there exists a linear map

$$\begin{aligned} V^{-1}(\mathbf{p}_f, R_f)\colon V(\mathbf{p}_f, R_f)(\Re^4) &\to \Re^4, \\ V(\mathbf{p}_f, R_f)(\mathbf{b}) &\mapsto \mathbf{b}. \end{aligned} \tag{70}$$

We call $V^{-1}(\mathbf{p}_f, R_f)$ the *inverse addition of velocities map.* In physical terms, it calculates the motion of each sensor relative to the object in response to a relative motion between the sensors that satisfies the kinematic constraints of no slippage.

The relationship between the relative motion of the fingers and the motion of the points of contact is as shown in Fig. 7. Importantly, this relation can be inverted. Given desired values for the velocities of the points of contact on the surfaces of the objects, $\dot{\mathbf{u}}_{2a}$ and $\dot{\mathbf{u}}_{2b}$, we can find the unique relative velocity $(\mathbf{v}_f, \mathbf{w}_f)$ that produces these velocities for the points of contact and satisfies the slippage constraints.

7. Application 3: Rolling a Sphere

Consider two fingers grasping a sphere of radius R with one point of contact for each finger. Assume that, to start, the points of contact are diametrically opposed. Recall the coordinate system for a subset of the sphere described in Example 1. Embed the sphere in $\Re^3$ so that in this coordinate system the two points of contact have u coordinates (latitudes) equal to zero (i.e., lie on the equator). If the points of contact move on the surface of the sphere according to $\dot{\mathbf{u}}_{2a} = \dot{\mathbf{u}}_{2b} = [0, \dot{\mathbf{v}}]^T$, then they will remain on the equator diametrically opposed. Hence, a grip is maintained. When the points of contact move thus and when both fingers rotate without slipping relative to the sphere, we say that the fingers are rolling the sphere.

Proposition 5 *The unique velocity of finger a relative to finger b that satisfies the kinematic constraints of no slippage and produces velocities for the points of contact of* $\dot{\mathbf{u}}_{2a} = \dot{\mathbf{u}}_{2b} = [0, \dot{\mathbf{v}}]^T$ *is*

$$\begin{bmatrix} \mathbf{v}_f \\ \mathbf{w}_f \end{bmatrix} = \begin{bmatrix} 2(I + R\tilde{\tilde{K}}_{1b}) \\ 0 \quad 0 \\ J(\tilde{\tilde{K}}_{1b} - K_{1a}) \\ 0 \quad 0 \end{bmatrix} \begin{bmatrix} \dot{v} \sin \psi_a \\ \dot{v} \cos \psi_a \end{bmatrix}, \tag{71}$$

where

$$\begin{aligned} J &= \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \\ R_{ab} &= \begin{bmatrix} \cos(\psi_a + \psi_b) & -\sin(\psi_a + \psi_b) \\ -\sin(\psi_a + \psi_b) & -\cos(\psi_a + \psi_b) \end{bmatrix}, \\ \tilde{\tilde{K}}_{1b} &= R_{ab} K_{1b} R_{ab}. \end{aligned} \tag{72}$$

Proof: Since there is no slippage, $v_{xa} = v_{ya} = v_{xb} = v_{yb} = 0$. Hence, the second contact equation yields

$$\begin{aligned} \begin{bmatrix} -\omega_{ya} \\ \omega_{xa} \end{bmatrix} &= (K_{1a} + \tilde{K}_{2a}) R_\psi M_{2a} \dot{\mathbf{u}}_{2a} \\ &= \left(K_{1a} + \frac{1}{R} I \right) \begin{bmatrix} -\dot{v} \sin \psi_a \\ -\dot{v} \cos \psi_a \end{bmatrix}, \end{aligned} \tag{73}$$

$$\begin{aligned} \begin{bmatrix} -\omega_{yb} \\ \omega_{xb} \end{bmatrix} &= (K_{1b} + \tilde{K}_{2b}) R_\psi M_{2b} \dot{\mathbf{u}}_{2b} \\ &= \left(K_{1b} + \frac{1}{R} I \right) \begin{bmatrix} -\dot{v} \sin \psi_b \\ -\dot{v} \cos \psi_b \end{bmatrix}. \end{aligned} \tag{74}$$

Observe that the position and orientation of finger a relative to finger b are given by

$$\mathbf{p}_f = \begin{bmatrix} 0 \\ 0 \\ 2R \end{bmatrix}, \quad R_f = \begin{bmatrix} R_{ab} & 0 \\ 0 & 1 \end{bmatrix}. \tag{75}$$

After substituting Eqs. (73)–(75) into Eq. (58) and performing algebraic simplification, we get Eq. (71).

8. Application Four: Fine Grip Adjustment

In this section we examine the problem of controlling two fingers with tactile-sensing capability to actively adjust their grip so as to locally optimize some criterion rating possible grips. The criterion we choose is as follows. Let $\phi_b = \cos^{-1}([0, 0, 1]\mathbf{p}_f)$ and $\phi_a = \cos^{-1}(-[0, 0, 1]R_f^T\mathbf{p}_f)$. Then the smaller the value of $\max(\phi_a, \phi_b)$ the better is the grip. To see why, recall from Eq. (56) that two-fingered grips are characterized by the condition $\max(\phi_a, \phi_b) < \tan^{-1}(\kappa_s)$. Hence the smaller the value of $\max(\phi_a, \phi_b)$, the larger is the error required for the grip to be lost.

We now investigate how ϕ_a and ϕ_b depend on the motion of the points of contact. Let $\mathbf{n}_a(t)$ and $\mathbf{n}_b(t)$ be the inward normals to the object at the points of contact at time t. Let $\mathbf{d}_{ba}(t)$ be the vector from the point of contact b to the point of contact a. Relative to $C_{l_{1b}}(t)$, the local coordinate frame for finger b, these vectors are

$$\mathbf{n}_a(t) = R_f(t)\begin{bmatrix}0\\0\\1\end{bmatrix}, \qquad \mathbf{n}_b(t) = \begin{bmatrix}0\\0\\1\end{bmatrix}, \qquad \mathbf{d}_{ba}(t) = \mathbf{p}_f(t). \tag{76}$$

So, $\phi_b = \cos^{-1}(\mathbf{n}_b \cdot \mathbf{d}_{ba})$ and $\phi_a = \cos^{-1}(-\mathbf{n}_a \cdot \mathbf{d}_{ba})$. Over the time interval Δt the points of contact traverse small arc lengths $\Delta\tilde{s}_{2a}$ and $\Delta\tilde{s}_{2b}$ across the surface of the object. To first-order approximation, relative to the coordinate frame $C_{l_{1b}}(t)$,

$$\mathbf{n}_a(t+\Delta t) = R_f(t)\begin{bmatrix}-\tilde{K}_{2a}\,\Delta\tilde{s}_{2a}\\1\end{bmatrix}, \qquad \mathbf{n}_b(t+\Delta t) = \begin{bmatrix}-\tilde{K}_{2b}\,\Delta\tilde{s}_{2b}\\1\end{bmatrix}, \tag{77}$$

$$\mathbf{d}_{ba}(t+\Delta t) = \mathbf{p}_f + R_f(t)\begin{bmatrix}\Delta\tilde{s}_{2a}\\0\end{bmatrix} - \begin{bmatrix}\Delta\tilde{s}_{2b}\\0\end{bmatrix}. \tag{78}$$

Since dot products are invariant under coordinate frame transformation,

$$\phi_a(t+\Delta t) = \cos^{-1}(-\mathbf{n}_a(t+\Delta t)\cdot\mathbf{d}_{ba}(t+\Delta t)), \tag{79}$$

$$\phi_b(t+\Delta t) = \cos^{-1}(\mathbf{n}_b(t+\Delta t)\cdot\mathbf{d}_{ba}(t+\Delta t)), \tag{80}$$

where $\mathbf{n}_a(t+\Delta t)$, $\mathbf{n}_b(t+\Delta t)$, and $\mathbf{d}_{ba}(t+\Delta t)$ are as given in Eqs. (77) and (78). We can think of $\phi_a(t+\Delta t)$ and $\phi_b(t+\Delta t)$ as functions of $\Delta\tilde{s}_{2a}$ and $\Delta\tilde{s}_{2b}$. Thus, we define the function

$$f_1(\Delta\tilde{s}_{2a}, \Delta\tilde{s}_{2b}) = \max(\phi_a(\Delta\tilde{s}_{2a}, \Delta\tilde{s}_{2b}), \phi_b(\Delta\tilde{s}_{2a}, \Delta\tilde{s}_{2b})), \tag{81}$$

which is a rating of the grip obtained from the present one by motion of the points of contact across the surface of the object through arc lengths $\Delta\tilde{s}_{2a}$ and $\Delta\tilde{s}_{2b}$.

We further observe that, according to the second contact equation (as given in Eq. (39)), the angles of rotation needed to produce the arc length traversals $\Delta\tilde{s}_{2a}$ and $\Delta\tilde{s}_{2b}$ are

$$\Delta\theta_a = \begin{bmatrix}-\Delta\theta_{ya}\\ \Delta\theta_{xa}\end{bmatrix} = (K_{1a} + \tilde{K}_{2a})\Delta\tilde{s}_{2a}, \qquad \Delta\theta_b = \begin{bmatrix}-\Delta\theta_{yb}\\ \Delta\theta_{xb}\end{bmatrix} = (K_{1b} + \tilde{K}_{2b})\Delta\tilde{s}_{2b}. \tag{82}$$

We can then define the function

$$f_2(\Delta\tilde{s}_{2a}, \Delta\tilde{s}_{2b}) = \|\Delta\theta_a(\Delta\tilde{s}_{2a})\| + \|\Delta\theta_b(\Delta\tilde{s}_{2b})\|, \tag{83}$$

which is a measure of the size of the motion of the fingers.

We can perform a hill-climbing search to locally optimize the grip based on the following iterative step.

1. Use tactile sensing to measure the position of the points of contact on the two fingers. With proprioceptive sensing, determine the positions and orientations of the fingers. Based on these measurements, compute $\mathbf{p}_f$ and R_f, the relative position and orientation of the local coordinate frames, and K_{1a} and K_{1b}, the curvature forms of the fingers.
2. Perform curvature experiments to find K_{2a} and K_{2b}, the curvature forms of the object at each point of contact (recall Section 4). Curvature experiments involve only motions of the finger relative to the object such that the finger is

rolling without slipping. Hence, they can be performed while grasping the object based on the analysis of Section 6.

3. Find the values of $\Delta\tilde{s}_{2a}$ and $\Delta\tilde{s}_{2b}$, such that $f_2(\Delta\tilde{s}_{2a}, \Delta\tilde{s}_{2b}) \le \delta$, that minimize $f_1(\Delta\tilde{s}_{2a}, \Delta\tilde{s}_{2b})$. The parameter $\delta > 0$ is the maximum step size. If there are multiple sets of $\Delta\tilde{s}_{2a}$ and $\Delta\tilde{s}_{2b}$ that provide a minimum for $f_1(\Delta\tilde{s}_{2a}, \Delta\tilde{s}_{2b})$, choose one that minimizes $f_2(\Delta\tilde{s}_{2a}, \Delta\tilde{s}_{2b})$.
4. (optional) If, for the chosen values of $\Delta\tilde{s}_{2a}$ and $\Delta\tilde{s}_{2b}$, $f_1(0, 0) - f_1(\Delta\tilde{s}_{2a}, \Delta\tilde{s}_{2b}) < \epsilon$, where $\epsilon > 0$ is an appropriately chosen parameter, then stop the iteration and maintain the present grip.
5. Move the points of contact through arc lengths $\Delta\tilde{s}_{2a}$ and $\Delta\tilde{s}_{2b}$ across the surface of the object by rotating the fingers without slipping relative to the object through angles $\Delta\theta_a$ and $\Delta\theta_b$ as given in Eq. (82). Substituting into Eq. (58) gives the unique relative motion of the fingers that accomplishes this.
6. Repeat.

9. Conclusion

Using concepts from differential geometry, I have derived a set of equations, called contact equations, that are a general description of the kinematics of contact between two rigid bodies. Because of their generality, the contact equations are potentially a powerful tool for analyzing any task that involves contact evolving in time. Based on these equations, I have examined the following applications for a single end-effector: (1) determining the curvature form of an unknown object at a point of contact, and (2) following the surface of an unknown object. I have also used the contact equations to examine the kinematics of grasp. Based on this analysis, I have investigated these applications for two end-effectors: (1) rolling a sphere between two arbitrarily shaped fingers, and (2) fine grip adjustment (i.e., having two fingers that grasp an object locally optimize their grip for maximum stability).

Experimental work to corroborate the theory has been hampered by lack of resources, although there have been some preliminary but promising experiments performed. I have implemented a contour-following algorithm similar to that examined in this paper. Its performance is detailed on Montana (1986). Also described is a set of experiments investigating the effect of compliance on the kinematics of contact (the theory of which is discussed in Montana (1986) but not here).

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