# State-Space and System Modeling

# State Space for Dynamical Systems

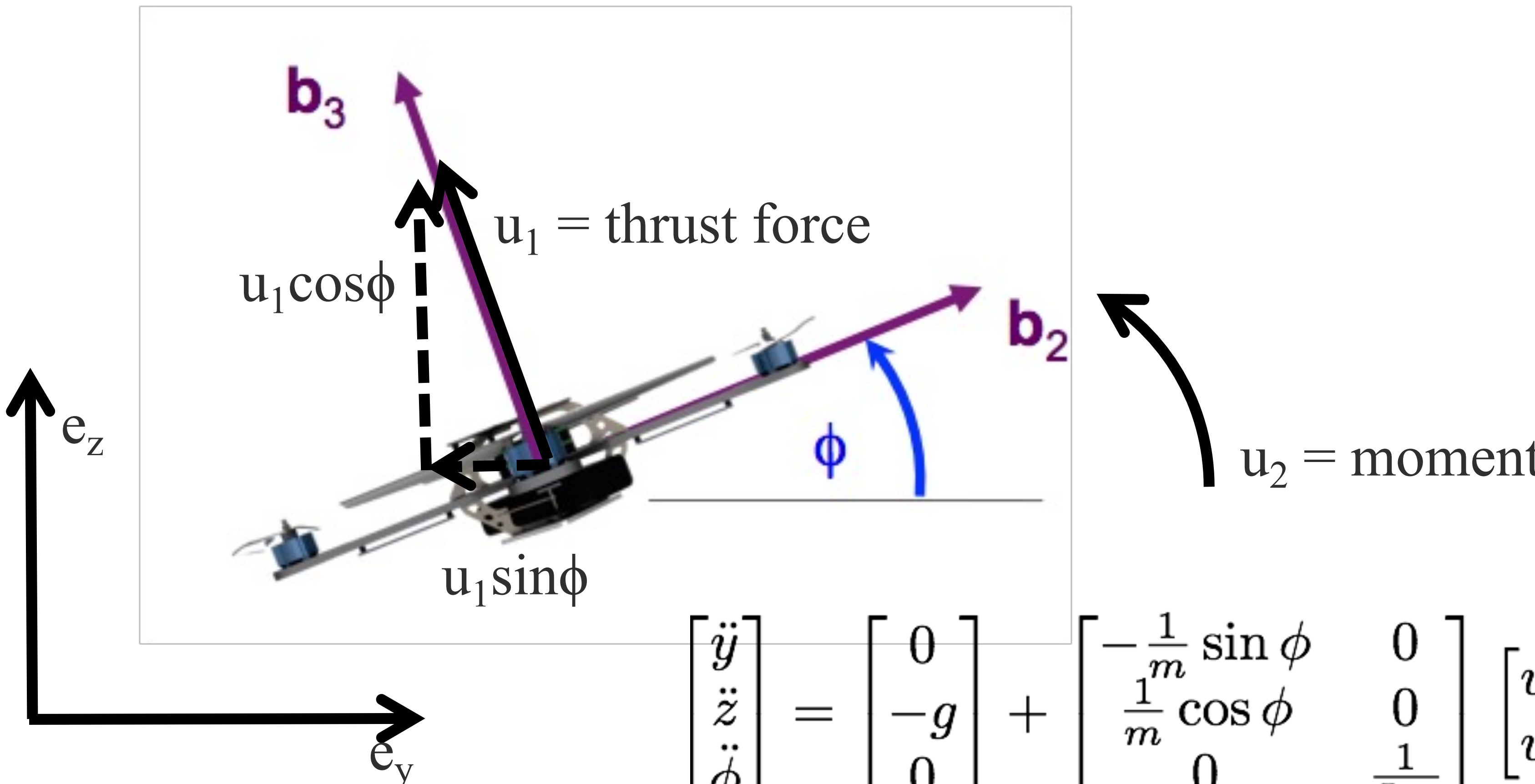
- State: $\mathbf{x} = [x_1 \;\; x_2 \;\; \cdots \;\; x_n]^{\mathrm{T}}$
- Robotic systems: The state typically includes the configuration $\mathbf{q}$ (position) and its derivative $\dot{\mathbf{q}}$ (velocity), that is

$$\mathbf{q} = \begin{bmatrix} q_1 \\ q_2 \\ \vdots \\ q_j \end{bmatrix}, \qquad \mathbf{x} = \begin{bmatrix} \mathbf{q} \\ \dot{\mathbf{q}} \end{bmatrix}$$

- The evolution of system's state over time is governed by a set of ordinary differential equations (ODEs).
- ODEs are often expressed in their equivalent state-space form

$$\dot{\mathbf{x}} = f(\mathbf{x}, \mathbf{u})$$

# Example: Planar Quadrotor

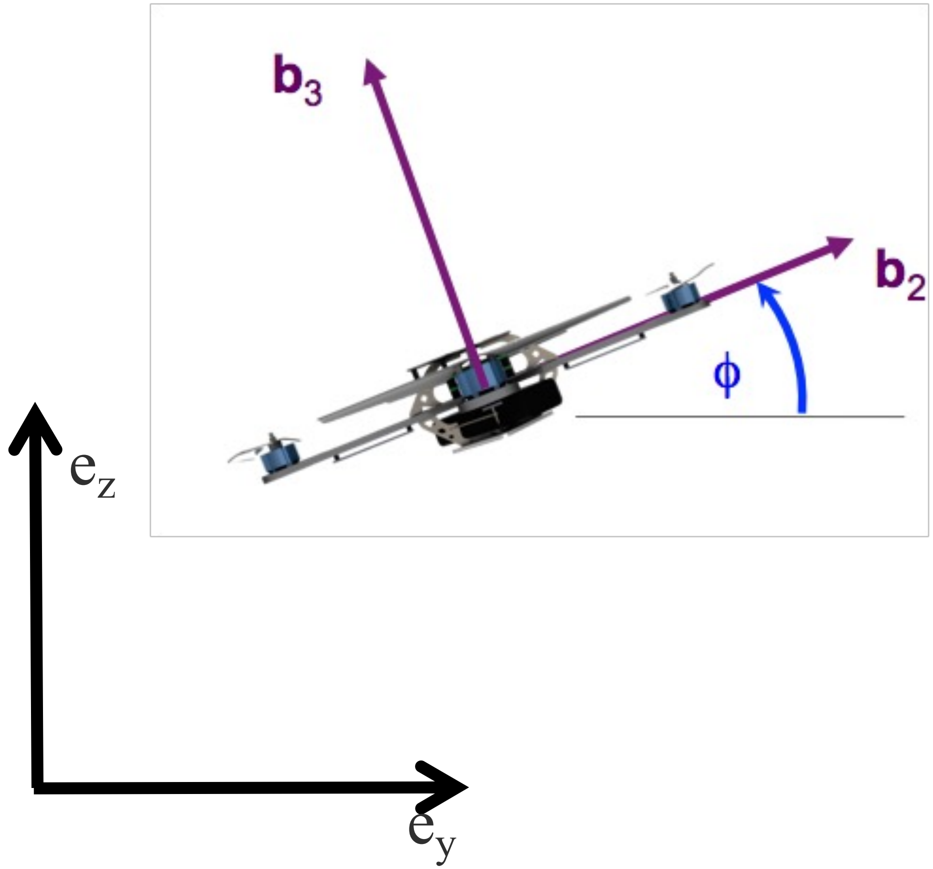



$$\begin{bmatrix} \ddot{y} \\ \ddot{z} \\ \ddot{\phi} \end{bmatrix} = \begin{bmatrix} 0 \\ -g \\ 0 \end{bmatrix} + \begin{bmatrix} -\frac{1}{m}\sin\phi & 0 \\ \frac{1}{m}\cos\phi & 0 \\ 0 & \frac{1}{I_{xx}} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

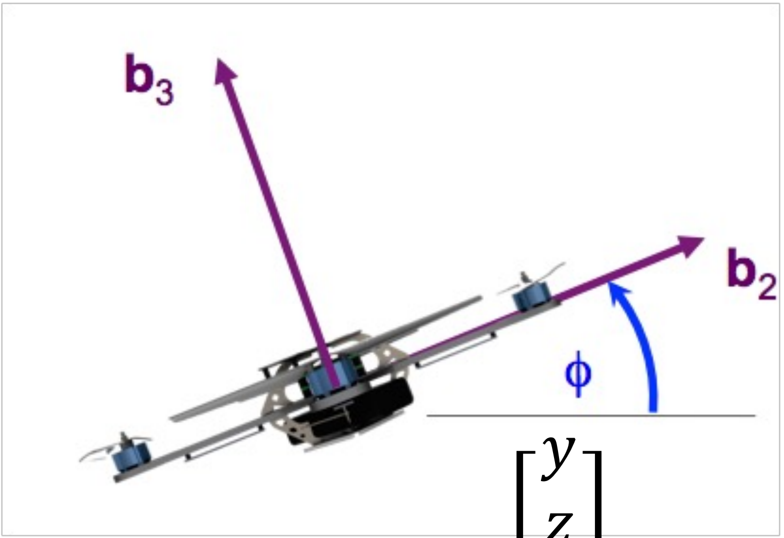
# State Space

State vector

$$\mathbf{q} = \begin{bmatrix} y \\ z \\ \phi \end{bmatrix}, \mathbf{x} = \begin{bmatrix} \mathbf{q} \\ \dot{\mathbf{q}} \end{bmatrix}$$



# Planar Quadrotor Model




$$\begin{bmatrix}\ddot{y}\\ \ddot{z}\\ \ddot{\phi}\end{bmatrix} = \begin{bmatrix}0\\ -g\\ 0\end{bmatrix} + \begin{bmatrix}-\frac{1}{m}\sin\phi & 0\\ \frac{1}{m}\cos\phi & 0\\ 0 & \frac{1}{I_{xx}}\end{bmatrix}\begin{bmatrix}u_1\\ u_2\end{bmatrix}$$

$$\mathbf{x} = \begin{bmatrix}\boldsymbol{q}\\ \dot{\boldsymbol{q}}\end{bmatrix} = \begin{bmatrix}y\\ z\\ \phi\\ \dot{y}\\ \dot{z}\\ \dot{\phi}\end{bmatrix}$$

$$\dot{\mathbf{x}} = \begin{bmatrix}x_4\\ x_5\\ x_6\\ 0\\ -g\\ 0\end{bmatrix} + \begin{bmatrix}0 & 0\\ 0 & 0\\ 0 & 0\\ -m^{-1}\sin x_3 & 0\\ m^{-1}\cos x_3 & 0\\ 0 & I_{xx}^{-1}\end{bmatrix}\begin{bmatrix}u_1\\ u_2\end{bmatrix}$$

$$\dot{\mathbf{x}} = f(\mathbf{x}, \mathbf{u})$$

# A Modeling Choice

# A Modeling Choice

# Main Steps to Build a State-Space Model

Given an ODE (for now of a single variable, $y(t)$)

- Isolate the $n^{\text{th}}$ highest derivative, $y^{(n)} = g(y, \dot{y}, \dots, y^{(n-1)}, \mathbf{u})$
- Set $x_1 = y(t),\ x_2 = \dot{y}(t),\ \dots,\ x_n = y^{(n-1)}(t)$
- Create state vector $\mathbf{x} = [x_1\ x_2\ \cdots\ x_n]^{\mathrm{T}} = \left[y\ \dot{y}\ \cdots\ y^{(n-1)}\right]^{\mathrm{T}}$
- Rewrite into a system of coupled first-order differential equations

$$\dot{x}_1 = \dot{y} = x_2$$

$$\dot{x}_2 = \ddot{y} = x_3$$

$$\dots$$

$$\dot{x}_n = y^{(n)} = g\left(y, \dot{y}, \dots, y^{(n-1)}, \mathbf{u}\right) = g(x_1, x_2, \dots, x_n, \mathbf{u})$$

# Main Steps to Build a State-Space Model

- Rewrite in matrix form

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} x_2 \\ x_3 \\ \vdots \\ g(x_1, x_2, \dots, x_n, \mathbf{u}) \end{bmatrix}$$

$$\dot{\mathbf{x}} = f(\mathbf{x}, \mathbf{u})$$

$\mathbf{x} \in \mathbb{R}^n, \mathbf{u} \in \mathbb{R}^m$

$n$ states $m$ inputs

- Note: A system is linear time-invariant (LTI) when

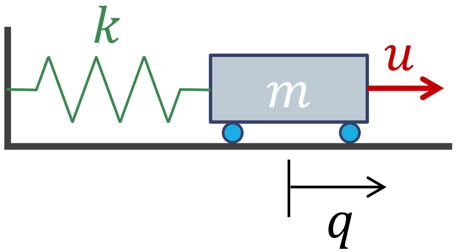
$$\dot{\mathbf{x}} = A\mathbf{x} + B\mathbf{u},$$

and A is an $n \times n$ constant matrix, and B is an $n \times m$ constant matrix.

# Example: Spring-Mass System

$$m\ddot{q}(t) + kq(t) = u(t)$$

Order of the system = 2




Rewrite $\ddot{q}(t) = \dfrac{u(t) - kq(t)}{m}$

State vector $\mathbf{x} = [x_1 \; x_2]^{\mathrm{T}} = [q \;\; \dot{q}]^{\mathrm{T}}$

Coupled equations $\dot{\mathrm{x}}_1 = x_2, \;\; \dot{x}_2 = \dfrac{u - kx_1}{m}$

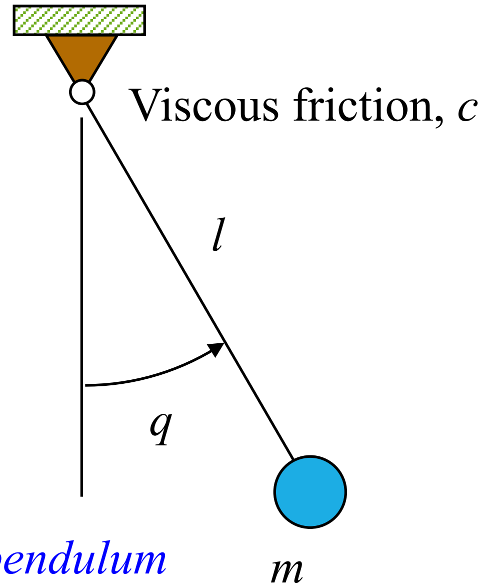
Then $\begin{bmatrix}\dot{x}_1 \\ \dot{x}_2\end{bmatrix} = \begin{bmatrix} x_2 \\ \dfrac{u - kx_1}{m}\end{bmatrix}$, or $\begin{bmatrix}\dot{x}_1 \\ \dot{x}_2\end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\dfrac{k}{m} & 0\end{bmatrix}\begin{bmatrix}x_1 \\ x_2\end{bmatrix} + \begin{bmatrix} 0 \\ \dfrac{1}{m}\end{bmatrix} u$

➢ Linear system

# Example: Damped Pendulum

Equation of motion

$$\ddot{q} + \frac{c}{ml^2}\dot{q} + \frac{g}{l}\sin q = 0$$




State space representation

$$x = \begin{bmatrix} q \\ \dot{q} \end{bmatrix}, \dot{x} = \begin{bmatrix} x_2 \\ -\frac{g}{l}\sin x_1 - \frac{c}{ml^2}x_2 \end{bmatrix}$$

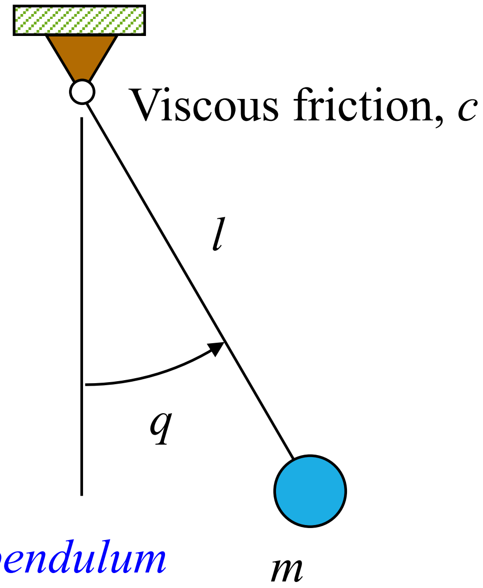
*simple pendulum*

- **Nonlinear** ⟹ Linearize around equilibria $\dot{\mathbf{x}} = \boxed{f(\mathbf{x}) \equiv 0}$

# Example: Damped Pendulum

Equation of motion

$$\ddot{q} + \frac{c}{ml^2}\dot{q} + \frac{g}{l}\sin q = 0$$




simple pendulum

State space representation

$$x = \begin{bmatrix} q \\ \dot{q} \end{bmatrix}, \dot{x} = \begin{bmatrix} x_2 \\ -\frac{g}{l}\sin x_1 - \frac{c}{ml^2}x_2 \end{bmatrix}$$

Equilibrium point(s)

$$x_{e,1} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, x_{e,2} = \begin{bmatrix} \pi \\ 0 \end{bmatrix}$$

Consider points near equilibria

$$\tilde{x} = \left(x - x_{e,i}\right)$$

# Equilibria

Consider a system with $n$ degrees of freedom

Let $q_e$ be a configuration at static equilibrium ($\dot{\mathbf{x}} = f(\mathbf{x}) \equiv 0$)

$$x(t_0) = \begin{bmatrix} q_e \\ \text{-- --} \\ 0 \end{bmatrix} \Rightarrow x(t > t_0) = \begin{bmatrix} q_e \\ \text{-- --} \\ 0 \end{bmatrix}$$

- An equilibrium point can be
  - Stable
  - Unstable
  - Critically stable (or neutrally stable)
- We are interested in the behavior of the system around equilibrium points.
- We linearize the system around equilibria!

# Linearization

- Given a nonlinear system $\dot{\mathbf{x}} = f(\mathbf{x}, \mathbf{u})$, $\mathbf{x}, f \in \mathbb{R}^n, \mathbf{u} \in \mathbb{R}^m$, derive an approximate linear system $\dot{\mathbf{x}} = A\mathbf{x} + B\mathbf{u}$ about an equilibrium point $(\mathbf{x}_e, \mathbf{u}_e)$.

- Taylor series expansion around equilibrium point:

$$f(\mathbf{x}_e + \Delta\mathbf{x}, \mathbf{u}_\mathrm{e} + \Delta\mathbf{u}) = f(\mathbf{x}_e, \mathbf{u}_e) + \left[\frac{\partial f}{\partial \mathbf{x}}\right]_{(\mathbf{x}_e, \mathbf{u}_e)} \Delta\mathbf{x} + \left[\frac{\partial f}{\partial \mathbf{u}}\right]_{(\mathbf{x}_e, \mathbf{u}_e)} \Delta\mathbf{u} + \mathrm{H.O.T.}$$

$$\dot{\mathbf{x}} + \Delta\dot{\mathbf{x}} \approx f(\mathbf{x}_e, \mathbf{u}_e) + \left[\frac{\partial f}{\partial \mathbf{x}}\right]_{(\mathbf{x}_e, \mathbf{u}_e)} \Delta\mathbf{x} + \left[\frac{\partial f}{\partial \mathbf{u}}\right]_{(\mathbf{x}_e, \mathbf{u}_e)} \Delta\mathbf{u}$$

$$\Longrightarrow \Delta\dot{\mathbf{x}} = A\Delta\mathbf{x} + B\Delta\mathbf{u}$$

Re-defining $\Delta\mathbf{x} \triangleq \mathbf{x}$, and $\Delta\mathbf{u} \triangleq \mathbf{u}$ yields $\dot{\mathbf{x}} = A\mathbf{x} + B\mathbf{u}$ with

$$A_{n\times n} = \left[\frac{\partial f}{\partial \mathbf{x}}\right]_{(\mathbf{x}_e, \mathbf{u}_e)} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_n} \end{bmatrix}_{(\mathbf{x}_e, \mathbf{u}_e)}, \quad B_{n\times m} = \left[\frac{\partial f}{\partial \mathbf{u}}\right]_{(\mathbf{x}_e, \mathbf{u}_e)} = \begin{bmatrix} \frac{\partial f_1}{\partial u_1} & \cdots & \frac{\partial f_1}{\partial u_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial u_1} & \cdots & \frac{\partial f_n}{\partial u_n} \end{bmatrix}_{(\mathbf{x}_e, \mathbf{u}_e)}$$

# Pendulum Linearization

$$\dot{\mathbf{x}} = \begin{bmatrix} x_2 \\ -\frac{g}{l}\sin x_1 - \frac{c}{ml^2}x_2 \end{bmatrix} = f(\mathbf{x})$$

Equilibrium point 1

$$\mathbf{x}_{e,1} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\dot{\mathbf{x}} \approx \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix}_{\mathbf{x}_{e,1}} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 \\ -\frac{g}{l} & -\frac{c}{ml^2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Equilibrium point 2

$$\mathbf{x}_{e,2} = \begin{bmatrix} \pi \\ 0 \end{bmatrix}$$

$$\dot{\mathbf{x}} \approx \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix}_{\mathbf{x}_{e,2}} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 \\ \frac{g}{l} & -\frac{c}{ml^2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

# Pendulum Stability

A linear system $\dot{x} = A\boldsymbol{x}$ is stable iff the real parts of all eigenvalues of $A$ are negative.

Equilibrium point 1

$$\mathbf{x}_{e,1} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\dot{\mathbf{x}} \approx \begin{bmatrix} 0 & 1 \\ -\frac{g}{l} & -\frac{c}{ml^2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\lambda = -\frac{c}{2ml^2} \pm \sqrt{\left(\frac{c}{2ml^2}\right)^2 - \frac{g}{l}}$$

**Stable**

**Marginally stable if $c = 0$**

Equilibrium point 2

$$\mathbf{x}_{e,2} = \begin{bmatrix} \pi \\ 0 \end{bmatrix}$$

$$\dot{\mathbf{x}} \approx \begin{bmatrix} 0 & 1 \\ \frac{g}{l} & -\frac{c}{ml^2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\lambda = -\frac{c}{2ml^2} \pm \sqrt{\left(\frac{c}{2ml^2}\right)^2 + \frac{g}{l}}$$

**Unstable**