MEAM 620

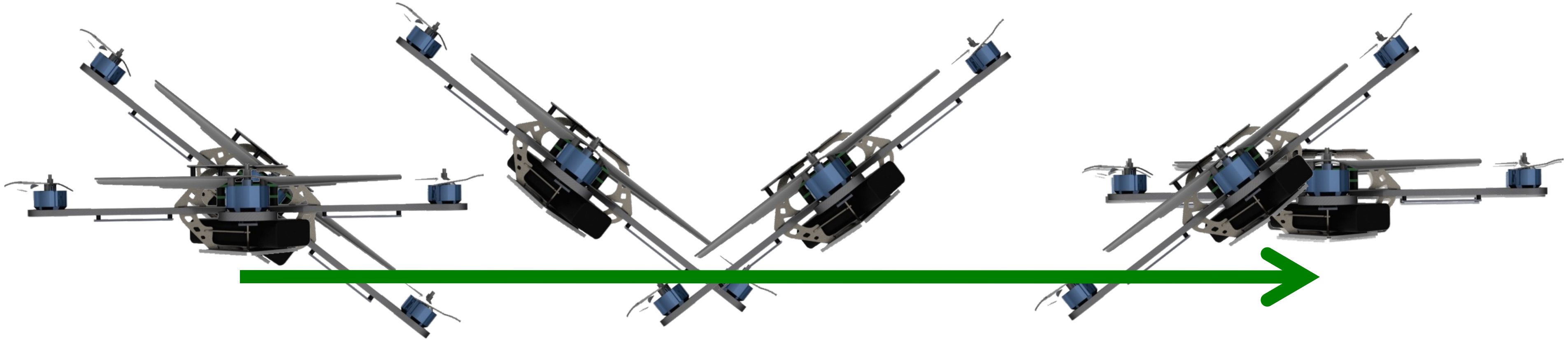
QUADROTOR DYNAMICS &
STATE-SPACE SYSTEM MODELING

Penn Engineering | GRASP Laboratory
General Robotics, Automation, Sensing & Perception Lab

What we'll Cover Today

- Newton-Euler equations of motion
  - Equations of motion for the quadrotor
- State-space system modeling & stability
  - State-space model for the quadrotor
- First assignment

Motivating Example

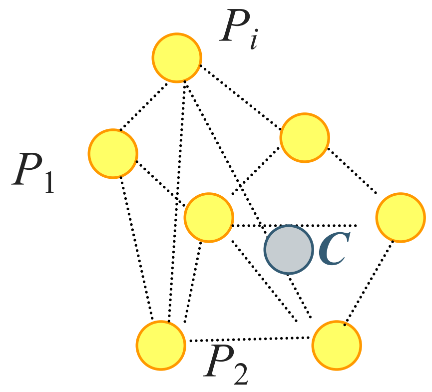


Newton-Euler Equations of Motion

Forces & Linear Momentum For Rigid Bodies

The rate of change of linear momentum $\boldsymbol{L}$ in an inertia frame A for a rigid body B equals the net applied force $\boldsymbol{F}$.

$$\mathbf{F} = \frac{{}^{A}d\mathbf{L}}{dt} \qquad \boldsymbol{L} = m\,{}^{A}\boldsymbol{v}^{C}$$



Where m is the total mass and ${}^{A}\boldsymbol{v}^{C}$ is the velocity of the center of mass, a point C located at $\boldsymbol{r}_C$.

center of mass: $$\mathbf{r}_c = \frac{1}{m}\sum_{i=1}^{N} m_i \mathbf{p}_i$$

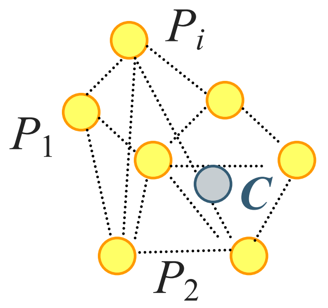
Net force: $$\mathbf{F} = \sum_{i=1}^{N} \mathbf{F}_i$$

then: $$\mathbf{F} = m\,\frac{{}^{A}d\,{}^{A}\mathbf{v}^{c}}{dt} \longrightarrow \mathbf{F} = m\,{}^{A}\boldsymbol{a}^{C}$$

The center of mass of a rigid body moves like a point mass m driven by a force F.

Moments & Angular Momentum for a Rigid Body

The rate of change of angular momentum $\boldsymbol{H}$ in an inertial frame A for a rigid body B relative to point C equals the net moment $\boldsymbol{M}$ about C due to applied forces relative to C.



$$\mathbf{M}_C^B = \frac{{}^A d\ {}^A\mathbf{H}_C^B}{dt} \qquad {}^A\mathbf{H}_C^B = \mathbf{I}_C \cdot {}^A\omega^B$$

Net moment on body B around point C from all external forces and torques

Angular momentum of body B about point C in frame A

inertia tensor

$$I_c = \begin{bmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{xy} & I_{yy} & I_{yz} \\ I_{xz} & I_{yz} & I_{zz} \end{bmatrix}$$

$$I_{xx} = \sum_{k=1}^{N} m_k (y_k^2 + z_k^2)$$

$$I_{xy} = -\sum_{k=1}^{N} m_k (x_k y_k)$$

Principal Axes

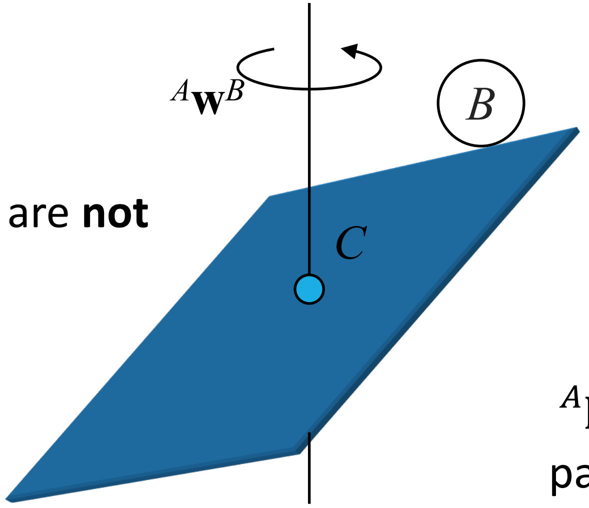
Principal axis of inertia

- $\mathbf{u}$ is a unit vector along a principal axis if $\mathbf{I}\mathbf{u}$ is parallel to $\mathbf{u}$
- You can always find 3 independent principal axes!
- Axes of symmetry are always principle axes.

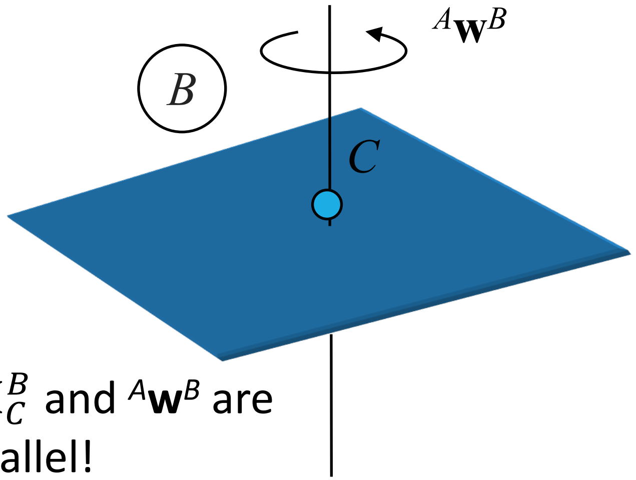
Principal moment of inertia

- The moment of inertia with respect to a principal axis, $\mathbf{u}^{\mathrm{T}}\mathbf{I}\mathbf{u}$, is called a principal moment of inertia

Physical interpretation



${}^{A}\mathbf{H}_{C}^{B}$ and ${}^{A}\mathbf{w}^{B}$ are **not** parallel!



${}^{A}\mathbf{H}_{C}^{B}$ and ${}^{A}\mathbf{w}^{B}$ are parallel!

(Recall ${}^{A}\mathbf{H}_{C}^{B} = \mathbf{I}_{C}\ {}^{A}\omega^{B}$)

Euler's Equations

What is the correct rotational analog to $\boldsymbol{F} = m\boldsymbol{a}$?

$$\mathbf{M}_C^B = \frac{{}^{A}d\ {}^{A}\mathbf{H}_C^B}{dt} = \frac{{}^{B}d\mathbf{H}_C^B}{dt} + \underbrace{{}^{A}\omega^{B} \times \mathbf{H}_C^B}_{\text{differentiating in a moving frame}}$$

$${}^{A}\mathbf{H}_C^B = \mathbf{I}_C \cdot {}^{A}\omega^{B}$$

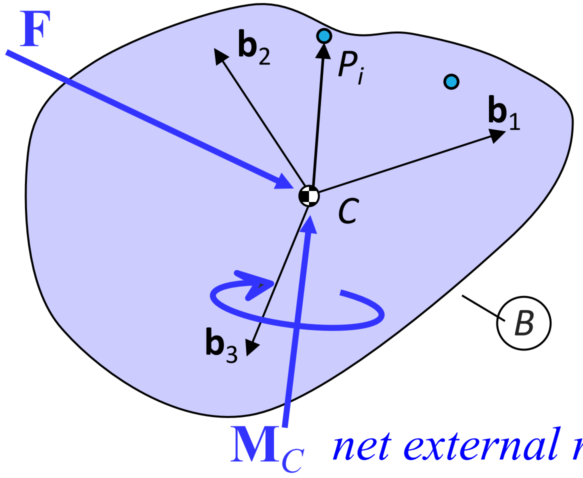
Not constant in inertial coordinates!

Using body-fixed coordinates, $\mathbf{I}_C$ is constant.

$$\frac{{}^{B}d\mathbf{H}_C^B}{dt} = \mathbf{I}_C \cdot {}^{A}\dot{\omega}^{B}$$

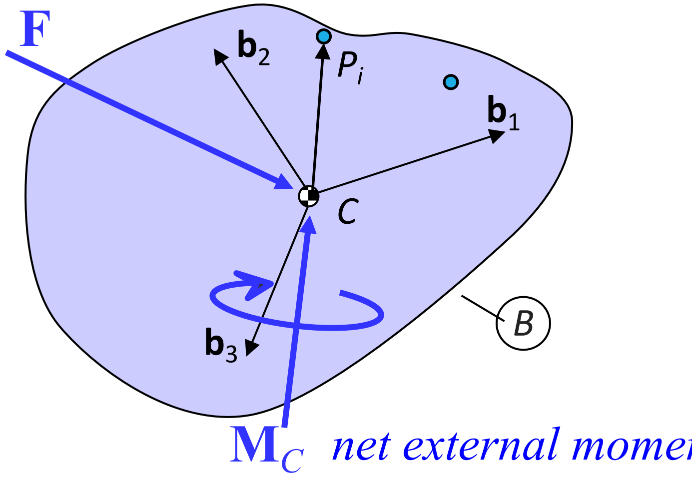
$${}^{A}\omega^{B} \times \mathbf{H}_C^B = {}^{A}\omega^{B} \times \mathbf{I}_C \cdot {}^{A}\omega^{B}$$

Euler's Equations: $\mathbf{I}_C\, {}^{A}\dot{\omega}^{B} + {}^{A}\omega^{B} \times \mathbf{I}_C\, {}^{A}\omega^{B} = \mathbf{M}_C^B$



Euler’s Equations

Define a body fixed frame with b1, b2, b3 all along principal axes.



From Euler’s Equation

$$\mathbf{I}_C \cdot {}^A\dot{\omega}^B + {}^A\omega^B \times \mathbf{I}_C \cdot {}^A\omega^B = \mathbf{M}_C^B$$

$$\begin{aligned} {}^A\boldsymbol{\omega}^B &= \omega_1 \boldsymbol{b}_1 + \omega_2 \boldsymbol{b}_2 + \omega_3 \boldsymbol{b}_3 \\ \boldsymbol{M}_C^B &= M_{C1} \boldsymbol{b}_1 + M_{C2} \boldsymbol{b}_2 + M_{C3} \boldsymbol{b}_3 \\ I_C &= \mathrm{diag}(I_{11}, I_{22}, I_{22}) \end{aligned}$$

Then, traditional matrix form for Euler’s Equations

$$\boxed{\begin{bmatrix} I_{11} & 0 & 0 \\ 0 & I_{22} & 0 \\ 0 & 0 & I_{33} \end{bmatrix} \begin{bmatrix} \dot{\omega}_1 \\ \dot{\omega}_2 \\ \dot{\omega}_3 \end{bmatrix} + \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix} \begin{bmatrix} I_{11} & 0 & 0 \\ 0 & I_{22} & 0 \\ 0 & 0 & I_{33} \end{bmatrix} \begin{array}{c} p \\ q \\ r \end{array} \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix} = \begin{bmatrix} M_{C,1} \\ M_{C,2} \\ M_{C,3} \end{bmatrix}}$$

Examples

Football thrown expertly.

Is the angular momentum constant?

$$\overset{0}{\cancel{\mathbf{M}_C^B}} = \frac{{}^A d\ {}^A\mathbf{H}_C^B}{dt}$$

Is the angular velocity constant?

$$\mathbf{I}_C\ {}^A\dot{\omega}^B + \overset{0}{\cancel{\left\{ {}^A\omega^B \times (\mathbf{I}_C\ {}^A\omega^B) \right\}}} = \overset{0}{\cancel{\mathbf{M}_C^B}}$$

Examples

Football thrown poorly.

Is the angular momentum constant?

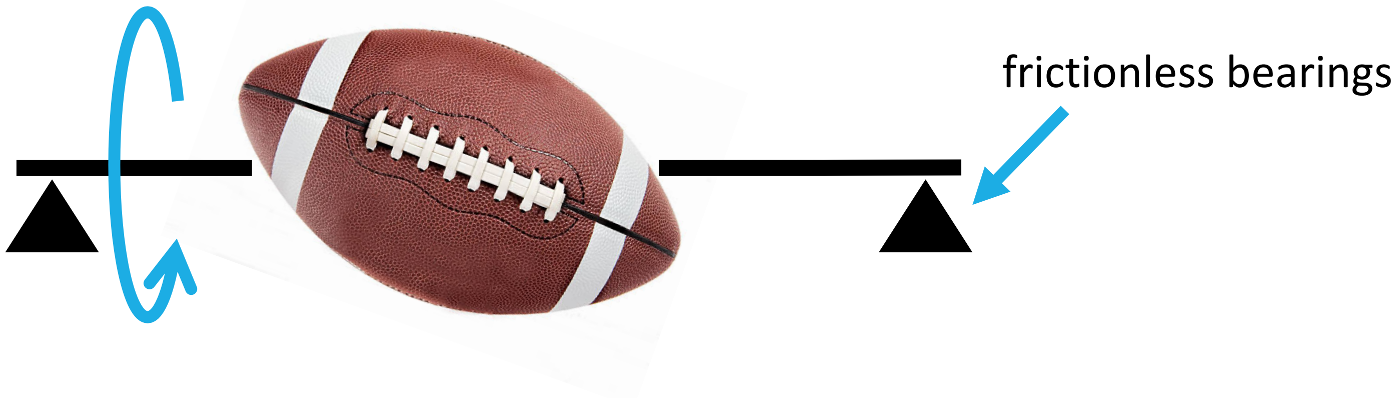
$$\cancelto{0}{\mathbf{M}_C^B} = \frac{{}^{A}d\ {}^{A}\mathbf{H}_C^B}{dt}$$

Is the angular velocity constant?

$$\mathbf{I}_C\ {}^{A}\dot{\omega}^B + \overbrace{\left\{ {}^{A}\omega^B \times (\mathbf{I}_C\ {}^{A}\omega^B) \right\}}^{\text{not zero!}} = \cancelto{0}{\mathbf{M}_C^B}$$

Examples

Football skewered on a stick.



Is the angular momentum constant?

not zero!

$$\mathbf{M}_C^B = \frac{{}^A d\ {}^A\mathbf{H}_C^B}{dt}$$

Is the angular velocity constant?

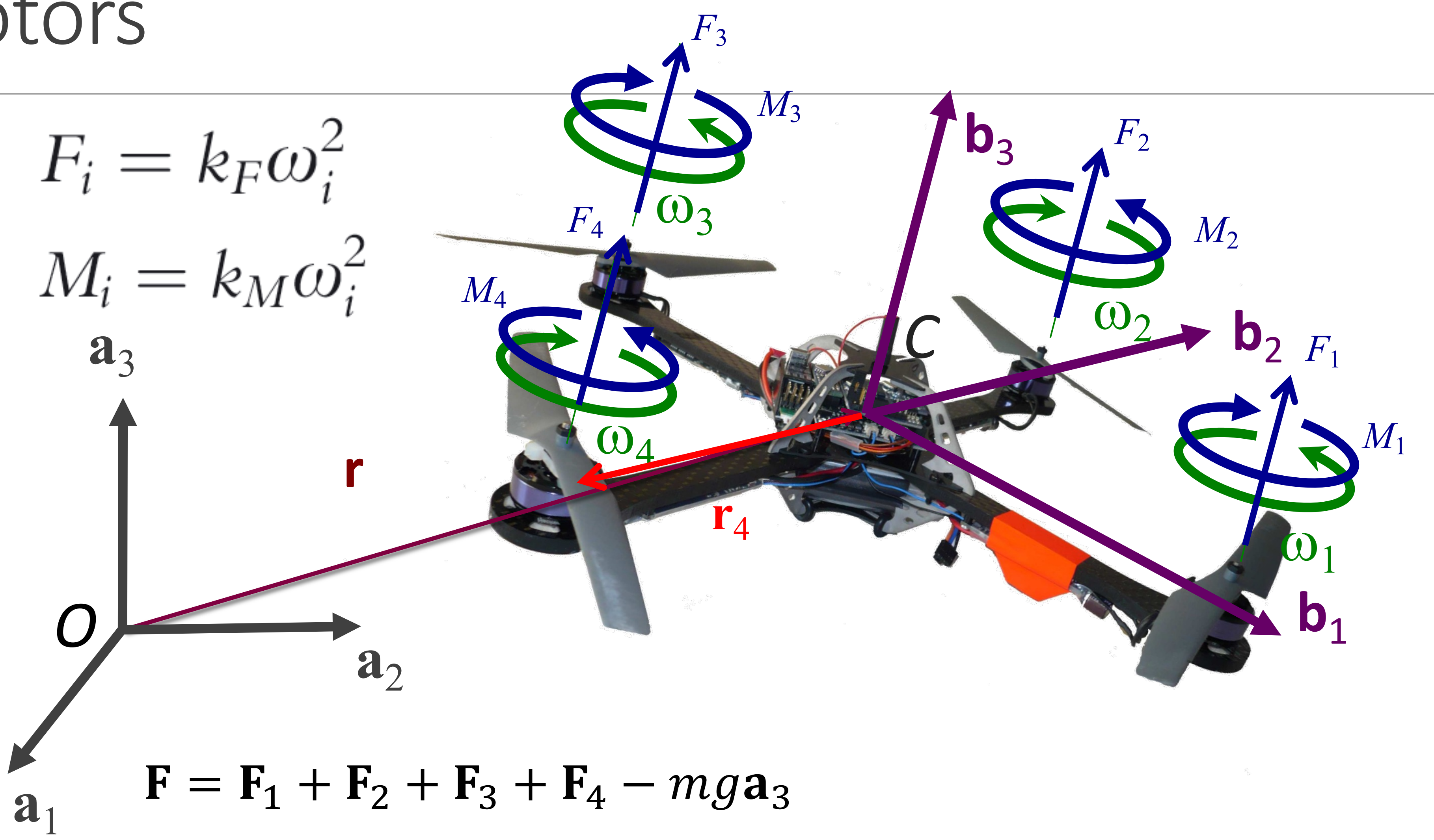
$$\mathbf{I}_C\ {}^A\dot{\omega}^B + {}^A\omega^B \times (\mathbf{I}_C \cdot {}^A\omega^B) = \mathbf{M}_C^B$$

Application to Quadrotors

Quadrotors

$$F_i = k_F \omega_i^2$$

$$M_i = k_M \omega_i^2$$



$$\mathbf{F} = \mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3 + \mathbf{F}_4 - mg\mathbf{a}_3$$

$$\mathbf{M} = (\mathbf{r}_1 \times \mathbf{F}_1 + \mathbf{r}_2 \times \mathbf{F}_2 + \mathbf{r}_3 \times \mathbf{F}_3 + \mathbf{r}_4 \times \mathbf{F}_4) \quad + (\mathbf{M}_1 + \mathbf{M}_2 + \mathbf{M}_3 + \mathbf{M}_4)$$

Newton-Euler Equations for a Quadrotor

$$^{A}\boldsymbol{\omega}^{B} = p\,\mathbf{b}_1 + q\,\mathbf{b}_2 + r\,\mathbf{b}_3$$

In inertial frame

$$m\ddot{\mathbf{r}} = \begin{bmatrix} 0 \\ 0 \\ -mg \end{bmatrix} + R \begin{bmatrix} 0 \\ 0 \\ \underbrace{\boxed{F_1 + F_2 + F_3 + F_4}}_{u_1} \end{bmatrix}$$

In body frame

$$I \begin{bmatrix} \dot{p} \\ \dot{q} \\ \dot{r} \end{bmatrix} = \underbrace{\boxed{\begin{bmatrix} L(F_2 - F_4) \\ L(F_3 - F_1) \\ M_1 - M_2 + M_3 - M_4 \end{bmatrix}}}_{\mathbf{u}_2} - \begin{bmatrix} p \\ q \\ r \end{bmatrix} \times I \begin{bmatrix} p \\ q \\ r \end{bmatrix}$$

Newton-Euler Equations for a Quadrotor

Recall that $\mathbf{F}_i = k_F \omega_i^2$ and $\mathbf{M}_i = k_M \omega_i^2$

Let $\gamma = \frac{k_M}{k_F} = \frac{\mathbf{M}_i}{\mathbf{F}_i} \Leftrightarrow \mathbf{M}_i = \gamma \mathbf{F}_i$

$$I \begin{bmatrix} \dot{p} \\ \dot{q} \\ \dot{r} \end{bmatrix} = \begin{bmatrix} L(F_2 - F_4) \\ L(F_3 - F_1) \\ M_1 - M_2 + M_3 - M_4 \end{bmatrix} - \begin{bmatrix} p \\ q \\ r \end{bmatrix} \times I \begin{bmatrix} p \\ q \\ r \end{bmatrix}$$

$$I \begin{bmatrix} \dot{p} \\ \dot{q} \\ \dot{r} \end{bmatrix} = \underbrace{\boxed{\begin{bmatrix} 0 & L & 0 & -L \\ -L & 0 & L & 0 \\ \gamma & -\gamma & \gamma & -\gamma \end{bmatrix} \begin{bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \end{bmatrix}}}_{\mathbf{u}_2} - \begin{bmatrix} p \\ q \\ r \end{bmatrix} \times I \begin{bmatrix} p \\ q \\ r \end{bmatrix}$$

Inputs

Putting everything together, we have inputs:

$$\mathbf{u} = \begin{bmatrix} \underline{u_1} \\ \\ \boldsymbol{u}_2 \\ \\ \end{bmatrix} = \begin{bmatrix} \underline{\text{thrust}} \\ \text{moment about } x \\ \text{moment about } y \\ \text{moment about } z \end{bmatrix}$$

$$\mathbf{u} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & L & 0 & -L \\ -L & 0 & L & 0 \\ \gamma & -\gamma & \gamma & -\gamma \end{bmatrix} \begin{bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \end{bmatrix} \qquad \mathrm{F}_i = k_F \omega_i^2$$

Note: All quantities are in the body frame!