Lecture Notes for A Mathematical Introduction to Robotic Manipulation

By

Z.X. Li* and Y.Q. Wu#

*Dept. of ECE, Hong Kong University of Science & Technology

#School of ME, Shanghai Jiaotong University

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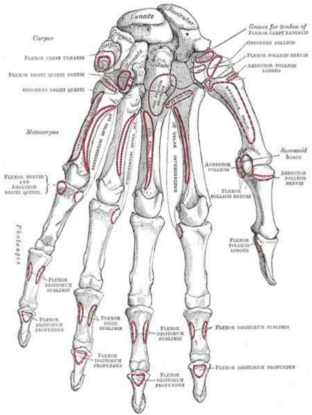
Chapter 5 Multifingered Hand

5.1 Introduction

□ *Hand function:*

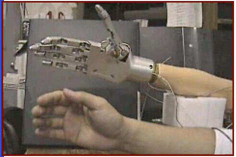
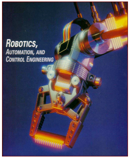


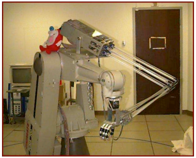
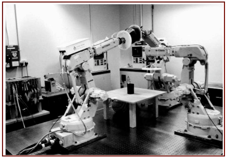
- Hand function:
  - Interface with external world
- Hand operation:
  - Grasping
  - Dextrous manipulation
  - Fine manipulation
  - Exploration

5.1 Introduction

☐ *History of hand design:*

- Prosthetic devices (1509)
- Dextrous end-effectors
- Multiple manipulator/agents coordination
- Human hand study

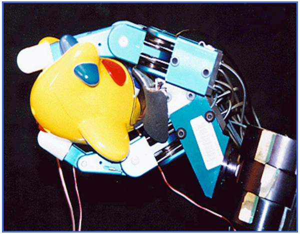



5.1 Introduction

□ *Hand Design Issues:*

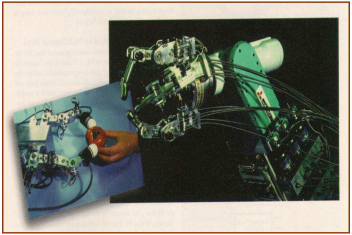
- Mechanical systems
- Sensor/actuators
- Control hardware

□ *A Sample List of Hand Prototypes:*



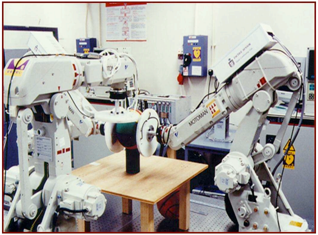
The salisbury Hand (1982)   The Utah-MIT Hand (1987)

5.1 Introduction

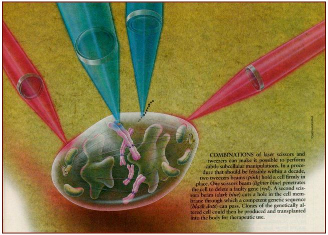


Toshiba Hand (Japan)

DLR hand (Germany, 1993)



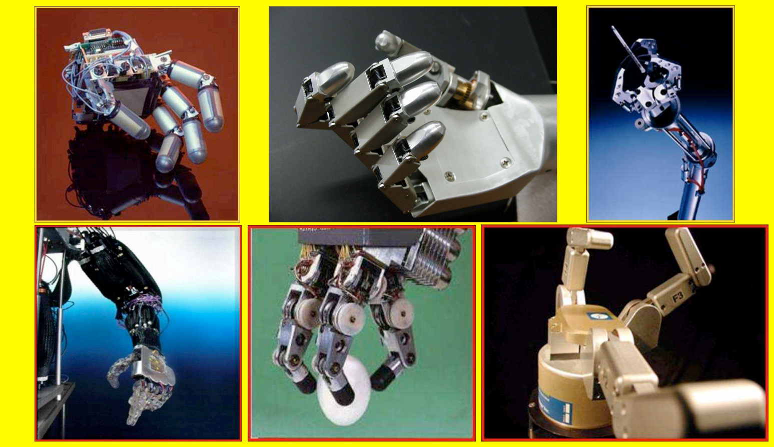
The HKUST Hand (1993)



Micro/Nano Hand

5.1 Introduction

⋄ *Example: More Hands*



5.1 Introduction

"The hand is indeed an instrument of creation par excellence."

Rodin (1840–1917)

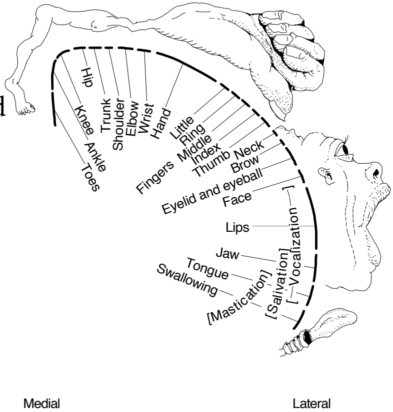
Rodin Hand (1898)

5.1 Introduction

□ *Lessons from Biological Systems:*

% of motor cortex for hand control:

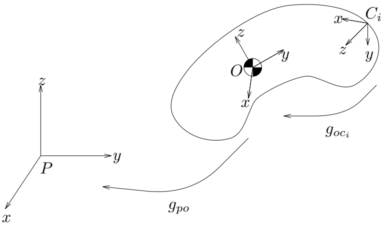
Human	30% ~ 40%
Monkey	20% ~ 30%
Dog	< 10%



Goal: A manipulation theory for robotic hand based on physical laws and rigorous mathematical models!

5.2 Grasp Statics

Contact Models:



1. Frictionless Point Contact (FPC)

$$F_i = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} x_i, x_i \geq 0$$

2. Point Contact with friction (PCWF)

$$F_i = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} x_i, x_i \in FC_i$$

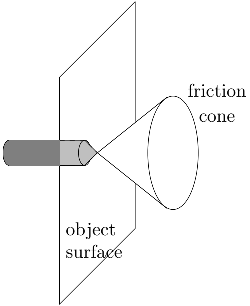
$$FC_i = \{x_i \in \mathbb{R}^3 : \sqrt{x_{i,1}^2 + x_{i,2}^2} \leq \mu_i x_{i,3}, x_{i,3} \geq 0\}$$

μ_i: Coulomb coefficient of friction

5.2 Grasp Statics

3 Soft finger contact (SFC)

$$F_i = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} x_i, x_i \in FC_i$$



Elliptic Model:

$$FC_i = \{x_i \in \mathbb{R}^4 | x_{i,3} \geq 0, \sqrt{\frac{1}{\mu_i^2}(x_{i,1}^2 + x_{i,2}^2) + \frac{1}{\mu_{it}^2} x_{i,4}^2} \leq x_{i,3}\}$$

Linear Model:

$$FC_i = \{x_i \in \mathbb{R}^4 | x_{i,3} \geq 0, \frac{1}{\mu_i}\sqrt{x_{i,1}^2 + x_{i,2}^2} + \frac{1}{\mu_{it}}|x_{i,4}| \leq x_{i,3}\}$$

μ_{it}: Torsional coefficient of friction

5.2 Grasp Statics

$F_i = B_i \cdot x_i, x_i \in FC_i, B_i \in \mathbb{R}^{6\times m_i}$: wrench basis

Property 1:

1. FC_i is a closed subset of $\mathbb{R}^{m_i}$, with nonempty interior.
2. $\forall x_1, x_2 \in FC_i, \alpha x_1 + \beta x_2 \in FC_i, \forall \alpha, \beta > 0$

□ *The Grasp Map:*

1. Single contact:

$$F_o = \mathrm{Ad}^T_{g_{oc_i}^{-1}} F_i = \underbrace{\begin{bmatrix} R_{oc_i} & 0 \\ \hat{p}_{oc_i} R_{oc_i} & R_{oc_i} \end{bmatrix}}_{G_i} B_i x_i, x_i \in FC_i, G_i = \mathrm{Ad}^T_{g_{oc_i}^{-1}} B_i$$

2. Multifingered grasp:

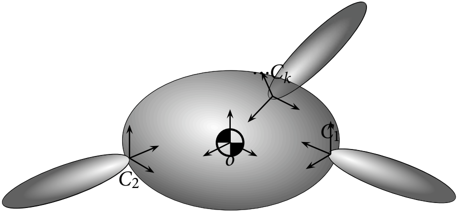
$$F_o = \sum_{i=1}^{k} G_i x_i = [\mathrm{Ad}^T_{g_{oc_1}^{-1}} B_1, \ldots, \mathrm{Ad}^T_{g_{oc_k}^{-1}} B_k] \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_k \end{bmatrix} \triangleq G \cdot x$$

$$x \in FC \triangleq FC_1 \times \cdots FC_k \subset \mathbb{R}^m, m = \sum_{i=1}^{k} m_i$$

5.2 Grasp Statics

Definition: Grasp

(G, FC) is called a grasp.



5.2 Grasp Statics

◇ *Example: Grasp map of frictionless point contact*

$$F_o = \begin{bmatrix} R_{c_i} & 0 \\ \hat{p}_{c_i} R_{c_i} & R_{c_i} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} x_i = \begin{bmatrix} n_{c_i} \\ p_{c_i} \times n_{c_i} \end{bmatrix} x_i, x_i \geq 0$$

$$\Rightarrow F_o = \begin{bmatrix} n_{c_1} & \cdots & n_{c_k} \\ p_{c_1} \times n_{c_1} & \cdots & p_{c_k} n_{c_k} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_k \end{bmatrix} = Gx$$

$F_o \in \mathbb{R}^6, x \geq 0.$

5.2 Grasp Statics

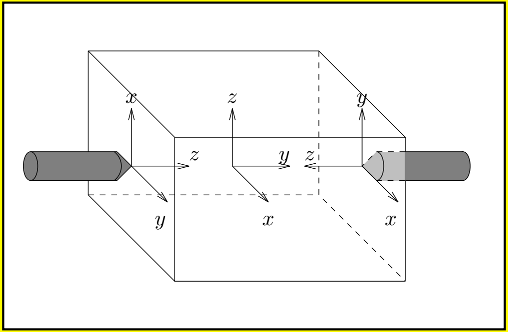
◇ *Example: Soft finger grasp of a box*

$$R_{c_1} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix},$$

$$p_{c_1} = \begin{bmatrix} 0 \\ -r \\ 0 \end{bmatrix},$$

$$R_{c_2} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix},$$

$$p_{c_2} = \begin{bmatrix} 0 \\ r \\ 0 \end{bmatrix},$$



$$G_i = \begin{bmatrix} R_{oc_i} & 0 \\ \hat{p}_{oc_i} R_{oc_i} & R_{oc_i} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

5.2 Grasp Statics

$$G = \begin{bmatrix} 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & r & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & r & 0 & 0 & -r & 0 & 0 & 0 \end{bmatrix}$$

$$x = \begin{bmatrix} x_{1,1} & x_{1,2} & x_{1,3} & x_{1,4} & x_{2,1} & x_{2,2} & x_{2,3} & x_{2,4} \end{bmatrix} \in \mathbb{R}^8$$

$$FC = FC_1 \times FC_2$$

$$FC_i = \left\{ x_i \in \mathbb{R}^4 \,\middle|\, \sqrt{\frac{1}{\mu_i}(x_{i,1}^2 + x_{i,2}^2) + \mu_{it} x_{i,4}^2} \leq x_{i,3}, x_{i,3} \geq 0 \right\}, i = 1, 2$$

5.2 Grasp Statics

□ *Friction Cone Representation:*

$$\text{FPC} \quad P_i(x_i) = [x_i]$$

$$\text{PCWF} \quad P_i(x_i) = \begin{bmatrix} \mu_i x_{i,3} + x_{i,1} & x_{i,2} \\ x_{i,2} & \mu_i x_{i,3} - x_{i,1} \end{bmatrix}$$

$$\text{SFC} \quad P_i(x_i) = \begin{bmatrix} x_{i,3} + x_{i,1}/\mu_i & x_{i,2}/\mu_i - j x_{i,4}/\mu_{it} \\ x_{i,2}/\mu_i + j x_{i,4}/\mu_{it} & x_{i,3} - x_{i,1}/\mu_i \end{bmatrix}, j^2 = -1$$

$$P(x) \triangleq \text{Diag}(P_1(x_i), \ldots, P_k(x_k)) \in \mathbb{R}^{\bar{m} \times \bar{m}} \; (\bar{m} = 2k)$$

$$x = (x_1, \ldots, x_m)^T = (x_1^T, \ldots, x_k^T) \in \mathbb{R}^m$$

5.2 Grasp Statics

Property 2:

1. $P_i = P_i^T, P = P^T$
2. $x_i \in FC_i$ (or $x \in FC$) $\Leftrightarrow P_i(x_i) \geq 0$ (or $P(x) \geq 0$)
3. $x_i \in FC_i \Leftrightarrow P_i(x_i) = \underbrace{S_i^0}_{=0} + \sum_{j=1}^{m_i} S_i^j x_{i,j} \geq 0, {S_i^j}^T = S_i^j, j = 1, \ldots, m_i$

   $x \in FC \Leftrightarrow P(x) = \underbrace{S_0}_{=0} + \sum_{i=1}^{m} S_i x_i \geq 0, S_i^T = S_i, i = 1, \ldots, m$
4. Let $Q(x) = \underbrace{S_0}_{=0} + \sum_{i=1}^{m} x_i S_i, S_i^T = S_i$, then

   $A_Q = \{x \in \mathbb{R}^m | Q(x) \geq 0\}$ and $B_Q = \{x \in \mathbb{R}^m | Q(x) > 0\}$ are both convex.

5.2 Grasp Statics

Proof of Property 2:

For PCWF,

$$\sigma(P_i) = \mu_i x_{i,3} \pm \sqrt{x_{i,1}^2 + x_{i,2}^2}$$

Thus,

$$\begin{aligned} x_i \in FC_i &\Leftrightarrow x_{i,3} \geq 0, \sqrt{x_{i,1}^2 + x_{i,2}} \leq \mu_i x_{i,3} \\ &\Leftrightarrow \sigma(P_i) \geq 0 \\ &\Leftrightarrow P_i \geq 0 \end{aligned}$$

Furthermore,

$$P_i(x_i) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} x_{i,1} + \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} x_{i,2} + \begin{bmatrix} \mu_i & 0 \\ 0 & \mu_i \end{bmatrix} x_{i,3} \qquad \square$$

Exercise: Verify this for SFC.

5.2 Grasp Statics

□ *Force Closure:*

Definition:
A grasp (G, FC) is force closure if $\forall F_o \in \mathbb{R}^p$ $(p = 3 \text{ or } 6)$, $\exists x \in FC$, s.t. $Gx = F_o$

Problem 1: Force-closure Problem
Determine if a grasp (G, FC) is force-closure or not.

Problem 2: Force Feasibility Problem
Given $F_o \in \mathbb{R}^p$, $p = 3$ or 6, determine if there exists $x \in FC$ s.t. $Gx = F_o$

Problem 3: Force Optimization Problem
Given $F_o \in \mathbb{R}^p$, $p = 3$ or 6, find $x \in FC$ s.t. $Gx = F_o$ and x minimizes $\Phi(x)$.

5.2 Grasp Statics

Definition: internal force

$x_N \in FC$ is an internal force if $Gx_N = 0$ or $x_N \in (\ker G \cap FC)$.

Property 3: (G, FC) is force closure iff $G(FC) = \mathbb{R}^p$ and $\exists x_N \in \ker G$ s.t. $x_N \in \mathrm{int}(FC)$.

5.2 Grasp Statics

Proof of Property 3:

Sufficiency:

For $F_o \in \mathbb{R}^p$, let x' be s.t. $F_o = Gx'$. Since $\lim_{\alpha\to\infty} \frac{x'+\alpha x_N}{\alpha} = x_N \in$ int(FC), there $\exists \alpha'$, sufficiently large, s.t.

$$\frac{x' + \alpha' x_N}{\alpha'} \in \text{int}(FC) \subset FC$$

$\Rightarrow x = x' + \alpha' x_N \in \text{int}(FC)$
$\Rightarrow Gx = Gx' = F_o$

Necessity:

Choose $x_1 \in \text{int}(FC)$ s.t. $F_o = Gx_1 \neq 0$, and choose $x_2 \in FC$ s.t. $Gx_2 = -F_o$. Define $x_N = x_1 + x_2$, $Gx_N = 0 \Rightarrow x_N \in \text{int}(FC)$ □

5.2 Grasp Statics

□ *Solutions of the force-closure and the force-feasibility problems (P1&P2):*

◊ *Review:*

Proposition: Linear matrix inequality (LMI) property

Given $Q(x) = S_0 + \sum_{l=1}^{m} x_l S_l$, where $S_l = S_l^T, l = 0, \ldots, m$, the sets $A_Q = \{x \in \mathbb{R}^m | Q(x) \geq 0\}$ and $B_Q = \{x \in \mathbb{R}^m | Q(x) > 0\}$ are convex.

◊ *Review:*

LMI Feasibility Problem:

Determine if the set A_Q or B_Q is empty or not.

5.2 Grasp Statics

◇ *Review:*

Convex constraints on $x \in \mathbb{R}^m$ (e.g., linear, (convex) quadratic, or matrix norm) can be transformed into LMI's.

- $x = \begin{bmatrix} x_1 \\ \vdots \\ x_m \end{bmatrix} \geq 0 \Leftrightarrow \text{diag}(x) \geq 0$
- For $A \in \mathbb{R}^{n\times m}, b \in \mathbb{R}^n, d \in \mathbb{R}$,

$$\|Ax+b\| \leq C^T x + d \Leftrightarrow \begin{bmatrix} (C^T x + d)I & Ax+b \\ (Ax+b)^T & C^T x + d \end{bmatrix} \geq 0$$

◇ *Review: Condition on force-closure*

1. $\text{Rank}(G) = 6$
2. $x_N \in \mathbb{R}^m, P(x_N) > 0$ and $Gx_N = 0$

Let $V = [v_1, \cdots, v_k]$ whose columns are the basis of $\ker G$

$$x_N = Vw, w \in \mathbb{R}^k$$
$$\tilde{P}(w) = P(Vw) = \sum_{l=1}^k w_l \tilde{S}_l, \tilde{S}_l = S_l^T, l = 1, \ldots, k$$

5.2 Grasp Statics

◇ *Review:*

Proposition: Given $Q(x) = S_0 + \sum_{i=1}^{m} x_l S_l$, where $S_l = S_l^T$, $l = 0, \ldots, m$. Let $x = Az + b$ where $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$, then $\tilde{Q}(z) = Q(Az + b) = \tilde{S}_0 + \sum_{l=1}^{n} z_l \tilde{S}_l$, where $\tilde{S}_l = \tilde{S}_l^T$, $l = 0, \ldots, n$

Theorem 1:

(G, FC) is force-closure iff $B_{\tilde{P}} = \{w \in \mathbb{R}^k | \sum_{l=1}^{k} w_l \tilde{S}_l > 0\}$ is non-empty.

- Force feasibility Problem

$$Gx = F_o \Rightarrow \quad x_o = \underbrace{G^\dagger}_{\text{generalized inverse}} F_o \text{ (may not lie in } FC)$$

$$x = G^\dagger F_o + Vw, \text{span}(V) = \ker G,$$

$$\bar{P}(w) = P(G^\dagger F_o + Vw)$$

$$= \bar{S}_0 + \sum_{l=1}^{k} \bar{S}_l w_l, \text{ where } \bar{S}_l = S_l^T, l = 0, \ldots, k$$

Theorem 2:

The force-feasibility problem for a given $F_o \in \mathbb{R}^6$ is solvable iff $A_{\bar{P}}(w) = \{w \in \mathbb{R}^k | \bar{S}_0 + \sum_{l=1}^{k} \bar{S}_l w_l > 0\}$ is non-empty.

◇ *Review: LMI Feasibility Problem as Optimization Problem*

$$\begin{aligned} Q \geq 0 \Leftrightarrow \exists t \leq 0 \text{ s.t. } Q + tI \geq 0 \\ \Leftrightarrow t \geq -\lambda_{\min}(Q) \\ \Leftrightarrow \lambda_{\min}(Q) \geq 0 \end{aligned}$$

Problem 2′: $\min t$

subject to$Q(x) + tI \geq 0$

If $t^* \leq 0$, then the LMI is feasible.

5.2 Grasp Statics

□ *Constructive force-closure for PCWF:*

$$G = \begin{bmatrix} n_{c_1} & \cdots & n_{c_k} \\ p_{c_1} \times n_{c_1} & \cdots & p_{c_k} \times n_{c_k} \end{bmatrix}, FC = \{x \in \mathbb{R}^k | x_i \geq 0\}$$

$G(FC) = \mathbb{R}^6 \Leftrightarrow$ Positive linear combination of column of $G = \mathbb{R}^6$

Definition:

- $v_1, \ldots, v_k, v_i \in \mathbb{R}^p$ is positively dependent if $\exists \alpha_i > 0$ such that $\sum \alpha_i v_i = 0$
- $v_1, \ldots, v_k, v_i \in \mathbb{R}^n$ positively span $\mathbb{R}^n$ if $\forall x \in \mathbb{R}^n, \exists \alpha_i > 0$, such that $\sum \alpha_i v_i = x$

Definition:

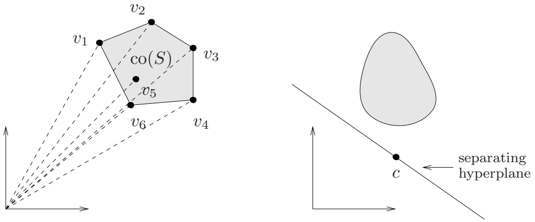
- A set K is convex if $\forall x, y \in K, \lambda x + (1-\lambda) y \in K, \lambda \in [0,1]$
- Given S=$v_1, \ldots, v_k, v_i \in \mathbb{R}^p$, the convex hull of S: $co(S) = \{v = \sum \alpha_i v_i, \sum \alpha_i = 1, \alpha_i \geq 0\}$

5.2 Grasp Statics

Property 4:

Let $G = \{G_1, \ldots, G_k\}$, the following are equivalent:

1. (G, FC) is force-closure.
2. The columns of G positively span $\mathbb{R}^p, p = 3, 6$
3. The convex hull of G_i contains a neighborhood of the origin.
4. There does not exist a vector $v \in \mathbb{R}^p, v \neq 0$ s.t. $\forall i = 1, \ldots, k, v \cdot G_i \geq 0$

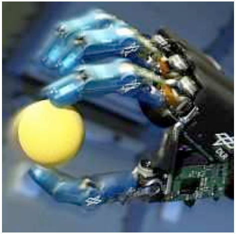


(a) (b)

5.3 Kinematics of Contact

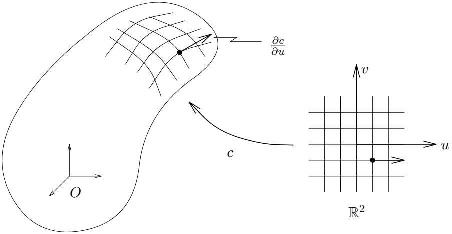
□ *Surface Model:*

$$c : U \subset \mathbb{R}^2 \to \mathbb{R}^3, c(U) \subset S$$



$$c_u = \frac{\partial c}{\partial u} \in \mathbb{R}^3$$

$$c_v = \frac{\partial c}{\partial v} \in \mathbb{R}^3$$



First Fundamental form: $I_p = \begin{bmatrix} c_u^T c_u & c_u^T c_v \\ c_v^T c_u & c_v^T c_v \end{bmatrix}$

5.3 Kinematics of Contact

Orthogonal Coordinates Chart: $c_u^T c_v = 0$ (assumption)

$$I_p = \begin{bmatrix} \|c_u\|^2 & 0 \\ 0 & \|c_v\|^2 \end{bmatrix} = M_p \cdot M_p$$

Metric tensor:

$$M_p = \begin{bmatrix} \|c_u\| & 0 \\ 0 & \|c_v\| \end{bmatrix}$$

Gauss map:

$$N : S \to s^2 : N(u, v) = \frac{c_u \times c_v}{\|c_u \times c_v\|} := n$$

2nd Fundamental form:

$$II_p = \begin{bmatrix} c_u^T n_u & c_u^T n_v \\ c_v^T n_u & c_v^T n_v \end{bmatrix}, n_u = \frac{\partial n}{\partial u}, n_v = \frac{\partial n}{\partial v}$$

5.3 Kinematics of Contact

Curvature tensor:

$$K_p = M_p^{-T} II_p M_p^{-1} = \begin{bmatrix} \frac{c_u^T n_u}{\|c_u\|^2} & \frac{c_u^T n_v}{\|c_u\|\|c_v\|} \\ \frac{c_v^T n_u}{\|c_u\|\|c_v\|} & \frac{c_v^T n_v}{\|c_v\|^2} \end{bmatrix}$$

Gauss frame:

$$[x, y, z] = \begin{bmatrix} \frac{c_u}{\|c_u\|} & \frac{c_v}{\|c_v\|} & n \end{bmatrix}, K_p = \begin{bmatrix} x^T \\ y^T \end{bmatrix} \begin{bmatrix} \frac{n_u}{\|c_u\|} & \frac{n_v}{\|c_v\|} \end{bmatrix}$$

Torsion form:

$$T_p = y^T \begin{bmatrix} \frac{x_u}{\|c_u\|} & \frac{x_v}{\|c_v\|} \end{bmatrix}$$

(M_p, K_p, T_p):Geometric parameter of the surface.

5.3 Kinematics of Contact

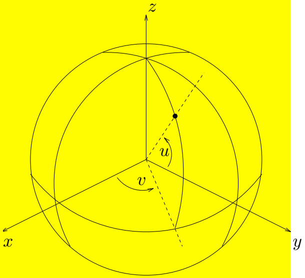
◇ *Example: Geometric parameters of a sphere in* $\mathbb{R}^3$

$$c(u,v) = \begin{bmatrix} \rho\cos u\cos v \\ \rho\cos u\sin v \\ \rho\sin u \end{bmatrix}$$

$$U = \{(u,v) \mid -\tfrac{\pi}{2} < u < \tfrac{\pi}{2}, -\pi < v < \pi\}$$

$$c_u = \begin{bmatrix} -\rho\sin u\cos v \\ -\rho\sin u\sin v \\ \rho\cos v \end{bmatrix}$$

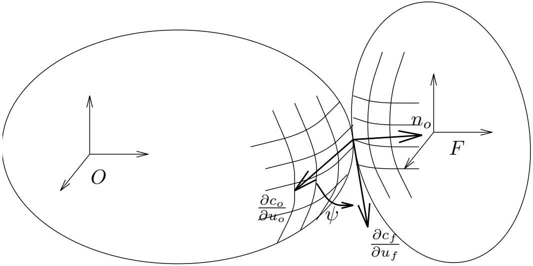
$$c_v = \begin{bmatrix} -\rho\cos u\sin v \\ \rho\cos u\cos v \\ 0 \end{bmatrix}$$



$$c_u^T c_v = 0$$

$$K = \begin{bmatrix} \frac{1}{\rho} & 0 \\ 0 & \frac{1}{\rho} \end{bmatrix}, M = \begin{bmatrix} \rho & 0 \\ 0 & \rho\cos u \end{bmatrix}, T = \begin{bmatrix} 0 & \frac{\tan v}{\rho} \end{bmatrix}$$

□ Gauss Frame:



$g_{oc}(t) \in SE(3)$

$V^b_{oc} = ?$

$$p_{oc}(t) = p(t) = c(\alpha(t)), R_{oc}(t) = [x(t), y(t), z(t)] = \begin{bmatrix} \frac{c_u}{\|c_u\|} & \frac{c_v}{\|c_v\|} & \frac{c_u \times c_u}{\|c_u \times c_v\|} \end{bmatrix}$$

$$v_{oc} = R^T_{oc}\dot{p}_{oc}(t) = \begin{bmatrix} x^T \\ y^T \\ z^T \end{bmatrix} \frac{\partial c}{\partial \alpha}\dot{\alpha} = \begin{bmatrix} x^T \\ y^T \\ z^T \end{bmatrix} \begin{bmatrix} c_u & c_v \end{bmatrix} \dot{\alpha} = \begin{bmatrix} \|c_u\| & 0 \\ 0 & \|c_v\| \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} M\dot{\alpha} \\ 0 \end{bmatrix}$$

$$\hat{\omega}_{oc} = R^T_{pc}\dot{R}_{oc} = \begin{bmatrix} x^T \\ y^T \\ z^T \end{bmatrix} \begin{bmatrix} \dot{x} & \dot{y} & \dot{z} \end{bmatrix} = \begin{bmatrix} 0 & x^T\dot{y} & x^T\dot{z} \\ y^T\dot{x} & 0 & y^T\dot{z} \\ z^T\dot{x} & z^T\dot{y} & 0 \end{bmatrix}$$

5.3 Kinematics of Contact

$$y^T \dot{x} = y^T [x_u \; x_v] \dot{\alpha} = TM\dot{\alpha}$$

$$\begin{bmatrix} x^T \dot{z} \\ y^T \dot{z} \end{bmatrix} = \begin{bmatrix} x^T \\ y^T \end{bmatrix} \dot{z} = \begin{bmatrix} x^T \\ y^T \end{bmatrix} \begin{bmatrix} n_u & n_v \end{bmatrix} \dot{\alpha} = KM\dot{\alpha}$$

$$\hat{\omega}_{oc} = \left[\begin{array}{cc|c} 0 & -TM\dot{\alpha} & KM\dot{\alpha} \\ TM\dot{\alpha} & 0 & \\ \hline -(KM\dot{\alpha})^T & & 0 \end{array} \right]$$

,

$$v_{oc} = \begin{bmatrix} M\dot{\alpha} \\ 0 \end{bmatrix}$$

5.3 Kinematics of Contact

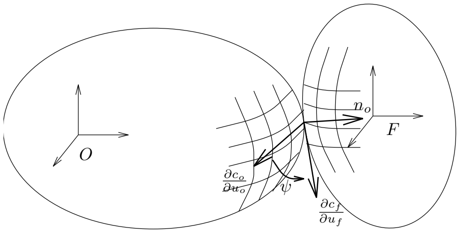
□ *Contact Kinematics:*

$$p_t \in S_0 \mapsto p_f(t) \in S_f$$

Local coordinate:

$$c_0 : U_0 \subset \mathbb{R}^2 \to S_0$$
$$c_f : U_f \subset \mathbb{R}^2 \to S_f$$

$$\alpha_0(t) = c_0^{-1}(p_0(t))$$
$$\alpha_f(t) = c_f^{-1}(p_f(t))$$

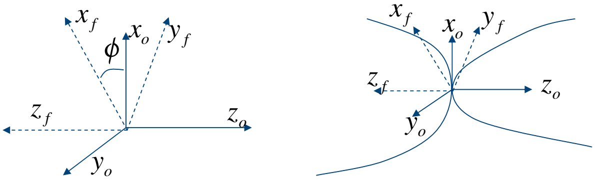


Angle of contact: ϕ
Contact coordinates: $\eta = (\alpha_f, \alpha_0, \phi)$

5.3 Kinematics of Contact

Rotation about the z-axis of C_o by $-\phi$ aligns the x axis of C_f with that of C_o

$$\Rightarrow R_{c_o c_f} = \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ -\sin\phi & -\cos\phi & 0 \\ 0 & 0 & -1 \end{bmatrix}, p_{c_o c_f} = 0 \in \mathbb{R}^3$$



5.3 Kinematics of Contact

Define $L_0(\tau)$:

At $\tau = t$, $L_0(\tau)$ coincide with the Gauss frame at $p_0(t)$.

$$L_f(\tau): \text{ coincide with } C_f(t) \text{ at } \tau = t$$

$$v_{l_0 l_f} = (v_x, v_y, v_z),$$

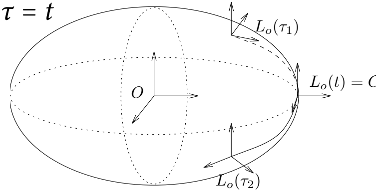
$$\omega_{l_0 l_f} = (\omega_x, \omega_y, \omega_z),$$

$$\begin{bmatrix} \omega_x \\ \omega_y \end{bmatrix}: \text{ Rolling velocities}$$

$$\begin{bmatrix} v_x \\ v_y \end{bmatrix}: \text{ Sliding velocities}$$

$$v_z: \text{ Linear velocity in the normal direction}$$

$$V_{l_0 l_f} = Ad_{g_{f l_f}} V_{of}: \text{ Velocity of the finger relative to the object}$$



5.3 Kinematics of Contact

Define:$\tilde{K}_0 = R_\phi K_0 R_\phi$:Curvature of O relative to C_f

$K_f + \tilde{K}_0$: Relative Curvature.

Theorem 3: Montana Equations of contact

$$\begin{cases} \dot{\alpha}_f = M_f^{-1}(K_f + \tilde{K}_o)^{-1}\left(\begin{bmatrix} -\omega_y \\ \omega_x \end{bmatrix} - \tilde{K}_o \begin{bmatrix} v_x \\ v_y \end{bmatrix}\right) \\ \dot{\alpha}_o = M_o^{-1} R (K_f + \tilde{K}_o)^{-1}\left(\begin{bmatrix} -\omega_y \\ \omega_x \end{bmatrix} + \tilde{K}_f \begin{bmatrix} v_x \\ v_y \end{bmatrix}\right)_\psi \\ \dot{\psi} = \omega_z + T_f M_f \dot{\alpha}_f + T_o M_o \dot{\alpha}_o \\ v_z = 0 \end{cases}$$

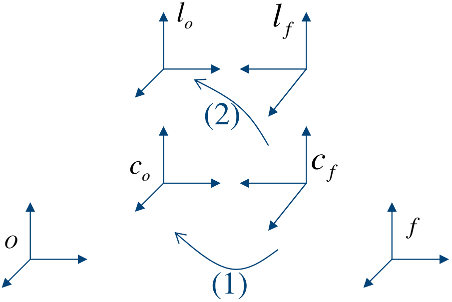
5.3 Kinematics of Contact

Proof of Theorem 3:

$$V_{fl_f} = 0$$

$$V_{fc_f} = Ad_{g^{-1}_{l_f c_f}} V_{fl_f} + V_{l_f c_f} = V_{l_f c_f}$$

$$V_{oc_o} = Ad_{g^{-1}_{l_o c_o}} V_{ol_o} + V_{l_o c_o} = V_{l_o c_o}$$



5.3 Kinematics of Contact

(1). At time t, $P_{l_f c_f} = 0$, $R_{l_f c_f} = I$ $\Rightarrow V_{l_o c_f} = V_{l_o l_f} + V_{l_f c_f}$

(2) $p_{c_o c_f} = 0 \Rightarrow V_{l_o c_f} = \begin{bmatrix} R^T_{c_o c_f} & 0 \\ 0 & R^T_{c_o c_f} \end{bmatrix} V_{l_o c_o} + V_{c_o c_f}$

$$\Rightarrow V_{l_o l_f} + V_{f c_f} = \begin{bmatrix} R^T_{c_o c_f} & 0 \\ 0 & R^T_{c_o c_f} \end{bmatrix} V_{o c_o} + V_{c_o c_f}$$

$p_{c_o c_f} = 0 \Rightarrow V_{c_o c_f} = 0$ $\qquad R_{c_o c_f} = \begin{bmatrix} R_\psi & 0 \\ 0 & -1 \end{bmatrix} \Rightarrow \omega_{c_o c_f} = \begin{bmatrix} 0 \\ 0 \\ \dot{\psi} \end{bmatrix}$

$V_{l_o l_f} = \begin{bmatrix} v_x \\ v_y \\ v_z \\ \omega_x \\ \omega_y \\ \omega_z \end{bmatrix}, V_{f c_f} = \begin{bmatrix} M_f \dot{\alpha}_f \\ 0 \end{bmatrix}$

$$\hat{\omega}_{f c_f} = \left[\begin{array}{cc|c} 0 & -T_f M_f \dot{\alpha}_f & K_f M_f \dot{\alpha}_f \\ T_f M_f \dot{\alpha}_f & 0 & \\ \hline -(K_f M_f \dot{\alpha}_f)^T & & 0 \end{array}\right]$$

5.3 Kinematics of Contact

$$v_{oc_o} = \begin{bmatrix} M_o\dot{\alpha}_o \\ 0 \end{bmatrix}, \quad \hat{\omega}_{oc_o} = \begin{bmatrix} 0 & -T_oM_o\dot{\alpha}_o & K_oM_o\dot{\alpha}_o \\ T_oM_o\dot{\alpha}_o & 0 & \\ -(K_oM_o\dot{\alpha}_o)^T & & 0 \end{bmatrix}$$

Linear component: $\begin{bmatrix} M_f\dot{\alpha}_f \\ 0 \end{bmatrix} + \begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix} = \begin{bmatrix} R_\psi M_o\dot{\alpha}_o \\ 0 \end{bmatrix}$

$$\begin{bmatrix} K_fM_f\dot{\alpha}_f \\ T_fM_f\dot{\alpha}_f \end{bmatrix} + \begin{bmatrix} \omega_y \\ -\omega_x \\ \omega_z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \dot{\psi} \end{bmatrix} - \begin{bmatrix} R_\psi K_oM_o\dot{\alpha}_o \\ T_oM_o\dot{\alpha}_o \end{bmatrix}$$

$\Rightarrow$ Theorem result □

Corollary: Rolling contact motion.

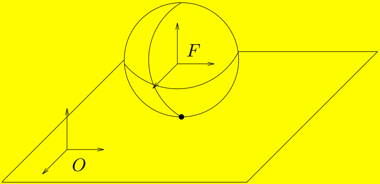
$$\dot{\alpha}_f = M_f^{-1}(K_f + \tilde{K}_o)^{-1}\begin{bmatrix} -\omega_y \\ \omega_x \end{bmatrix} \quad \dot{\alpha}_o = M_o^{-1}R_\psi (K_f + \tilde{K}_o)^{-1}\begin{bmatrix} -\omega_y \\ \omega_x \end{bmatrix} \quad \dot{\psi} = T_fM_f\dot{\alpha}_f + T_oM_o\dot{\alpha}_o$$

5.3 Kinematics of Contact

◇ *Example: A sphere rolling on a plane*

$$c_f(u,v) = \begin{bmatrix} \rho \cos u_f \cos v_f \\ \rho \cos u_f \sin v_f \\ \rho \sin u_f \end{bmatrix}$$

$$c_o(u,v) = \begin{bmatrix} u_o \\ v_o \\ 0 \end{bmatrix}$$



$$K_o = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, K_f = \begin{bmatrix} \frac{1}{\rho} & 0 \\ 0 & \frac{1}{\rho} \end{bmatrix}$$

$$M_o = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, M_f = \begin{bmatrix} \rho & 0 \\ 0 & \rho \end{bmatrix},$$

$$T_o = \begin{bmatrix} 0 & 0 \end{bmatrix}, T_f = \begin{bmatrix} 0 & -\frac{1}{\rho}\tan u_f \end{bmatrix}$$

5.3 Kinematics of Contact

$$\begin{bmatrix} \dot{u}_f \\ \dot{v}_f \\ \dot{u}_o \\ \dot{v}_o \\ \dot{\psi} \end{bmatrix} = \begin{bmatrix} 0 \\ \sec u_f \\ -\rho \sin\psi \\ -\rho\cos\psi \\ -\tan u_f \end{bmatrix} \omega_x + \begin{bmatrix} -1 \\ 0 \\ -\rho\cos\psi \\ \rho\sin\psi \\ 0 \end{bmatrix} \omega_y$$

$$\dot{\eta} = g_1(\eta) \underbrace{\omega_x}_{u_1(t)} + g_2(\eta) \underbrace{\omega_y}_{u_2(t)} \qquad (*)$$

Q:Given η_0, η_f, how to find a path u:o[0, T]→ $\mathbb{R}^2$ so that solution of (*) links η_0 to η_f?

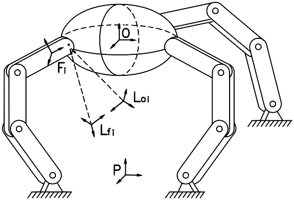
A question of nonholonomic motion planning!

5.4 Hand Kinematics

$$g_{po} = g_{pf_i}(\theta_i) g_{f_i l_{f_i}} g_{l_{f_i} l_{o_i}} g_{l_{o_i}}$$

$$V_{po} = \mathrm{Ad}_{g_{f_i o}^{-1}} V_{pf_i} + \mathrm{Ad}_{g_{l_{o_i} o}^{-1}} V_{l_{f_i} l_{o_i}}$$

$$\mathrm{Ad}_{g_{l_{o_i} o}} V_{po} = \mathrm{Ad}_{g_{f_i l_{o_i}}^{-1}} V_{pf_i} + V_{l_{f_i} l_{o_i}}$$



PCWF: $V_z = 0 \Rightarrow [0\ 0\ 1\ 0\ 0\ 0] V_{l_{f_i} l_{o_i}} = B_i^T V_{l_{f_i} l_{o_i}} = 0$

$$B_i^T \mathrm{Ad}_{g_{l_{o_i} o}} V_{po} = \mathrm{Ad}_{g_{f_i l_{o_i}}} \underbrace{V_{pf_i}}_{J_i(\theta_i)\dot{\theta}_i}$$

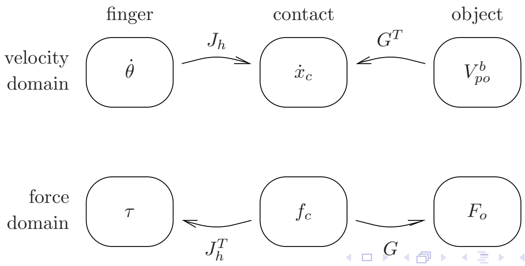
5.4 Hand Kinematics

$$\begin{bmatrix} B_1^T \mathrm{Ad}_{g_{l_{o_i} o}} \\ \vdots \\ B_k^T \mathrm{Ad}_{g_{l_{o_k} o}} \end{bmatrix} V_{po} = \begin{bmatrix} \mathrm{Ad}_{g^{-1}_{f_i l_{o_1}}} J_1(\theta_1) & & \\ & \ddots & \\ & & \mathrm{Ad}_{g^{-1}_{f_k l_{o_k}}} J_k(\theta_k) \end{bmatrix} \begin{bmatrix} \dot\theta_1 \\ \vdots \\ \dot\theta_k \end{bmatrix}$$

$$\boxed{G^T(\eta) V_{po} = J_h(\theta, x_0, \eta)\dot\theta}$$

$\theta = (\theta_1, \ldots, \theta_k) \in \mathbb{R}^n, n = \sum_{i=1}^{k} n_i, J_h \in \mathbb{R}^{m\times n}$: Hand Jacobian

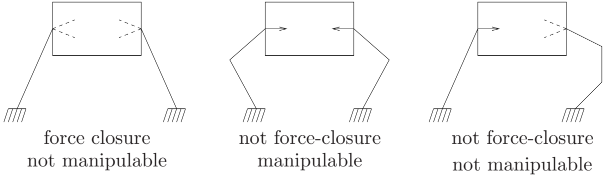
Definition: $\Omega = (G, FC, J_h)$ is called a multifingered grasp.



5.4 Hand Kinematics

Definition:
A multifingered grasp $\Omega = (G, FC, J_h)$ is manipulable at a configuration (θ, x_0) if for any object motion, $V_{po} \in \mathbb{R}^6$, $\exists \dot{\theta} \in \mathbb{R}^n$ s.t. $G^T V_{po} = J_h \dot{\theta}$.

Proposition:
Γ is manipulable at (θ, x°) iff $\operatorname{Im}(G^T) \subset \operatorname{Im}(J_h(\theta, x_0))$



5.4 Hand Kinematics

Table 5.4: Grasp properties.

Property	Definition	Description
Force-closure	Can resist any applied wrench	$G(FC) = \mathbb{R}^p$
Manipulable	Can accommodate any object motion	$\mathcal{R}(G^T) \subset \mathcal{R}(J_h)$
Internal forces	Contact forces f_N which cause no net object wrench	$f_N \in \mathcal{N}(G) \cap \text{int}(FC)$
Internal motions	Finger motions $\dot{\theta}_N$ which cause no object motion	$\dot{\theta}_N \in \mathcal{N}(J_h)$
Structural forces	Object wrench F_I which causes no net joint torques	$G^+ F_I \in \mathcal{N}(J_h^T)$

5.4 Hand Kinematics

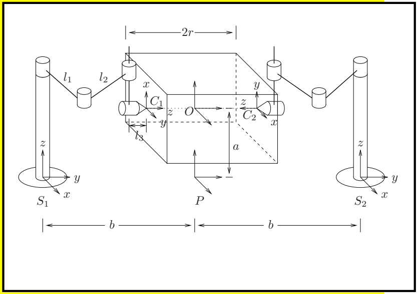
◇ *Example: Two SCARA fingers grasping a box*

Soft finger

$$G = \begin{bmatrix} 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ -r & 0 & 0 & 0 & 0 & +r & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & +r & 0 & 0 & -r & 0 & 0 & 0 \end{bmatrix}$$

$$B_{c_i} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$R_{po} = I, p_{po} = \begin{bmatrix} 0 \\ 0 \\ a \end{bmatrix}$$



5.4 Hand Kinematics

$$J_h = \begin{bmatrix} J_{11} & 0 \\ 0 & J_{22} \end{bmatrix}$$

$$J_{11} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ -b+r & -b+r+l_1c_1 & -b+r+l_1c_1+l_2c_{12} & 0 \\ 0 & l_1s_1 & l_1s_1+l_2s_{12} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

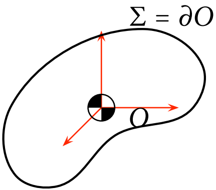
$$J_{22} = \begin{bmatrix} b-r & b-r+l_3c_3 & b-r+l_3c_3+l_4c_{34} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -l_3s_3 & -l_3s_3-l_4s_{34} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The grasp is not manipulable, as

$$G^T \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ -1 \end{bmatrix} \in \operatorname{Im}(J_h), \dot{\theta}_{N_1} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \dot{\theta}_{N_2} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

□ *Bounds on number of required contacts:*

Consider PCWF, let



$$\bigwedge(\Sigma) = \left\{ \begin{bmatrix} n_{c_i} \\ p_{c_i} \times n_{c_i} \end{bmatrix} \middle| c_i \in \Sigma \right\}$$

be the set of all wrenches, where n_{c_i} is inward normal.

Definition: Exceptional surface
The convex hull of $\bigwedge(\Sigma)$ does not contain a neighborhood of o in $\mathbb{R}^p$.

E.g. a Sphere or a circle.

Theorem 4: Caratheodory
If a set $X = (v_1, \ldots, v_k)$ positively spans $\mathbb{R}^p$, then $k \geq p + 1$

5.5 Grasp Planning

⇒ Lower bound on the number of fingers.

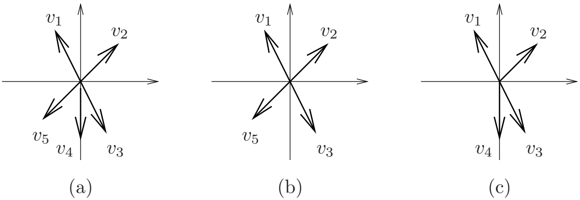


Figure 5.11: Sets of vectors which positively span $\mathbb{R}^2$.

Theorem 5: Steinitz

If $S \subset \mathbb{R}^p$ and $q \in \text{int}(co(S))$, then there exists $X = (v_1, \ldots, v_k) \subset S$ such that $q \in \text{int}(co(X))$ and $k \leq 2p$.

⇒ upper bound on the number of minimal fingers.

Table 5.3: Lower bounds on the number of fingers required to grasp an object.

Space	Object type	Lower	Upper	FPC	PCWF	SF
Planar	Exceptional	4	6	n/a	3	3
($p = 3$)	Non-exceptional			4	3	3
Spatial	Exceptional	7	12	n/a	4	4
($p = 6$)	Non-exceptional			12	4	4
	Polyhedral			7	4	4

□ *Constructing force-closure grasps:*

Theorem 1: Planar antipodal grasp

A planar grasp with two point contacts with friction is force-closure iff the line connecting the contact point lies inside both friction cones.

5.5 Grasp Planning

Theorem 2: Spatial antipodal grasps

A spatial grasp with two soft-finger contacts is force-closure iff the line connecting the contact points lies inside both friction cones.

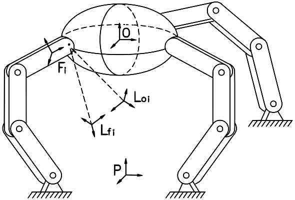
Problem 4: Optimal Grasp Synthesis

Plan a set of contact points on the object so that (G, FC) is force closure and optimal in some sense.

- ***Idea: construct a quality function:***

$$\psi : \alpha = \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_k \end{bmatrix} \in \mathbb{R}^{2k} \to \mathbb{R}$$

with computable gradient such that the optimal solution of ψ is also force closure.

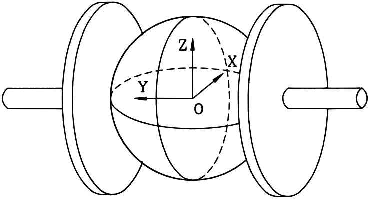


Grasp quality functions:

1. Two-finger grasps (Hong et al 90 & Chen and Burdick 93)

$$E = \frac{1}{2}\left\|X(\alpha_{o1}) - X(\alpha_{o2})\right\|^2$$

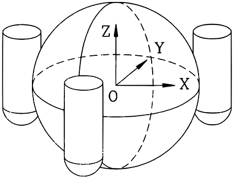
Solution: antipodal grasp



5.5 Grasp Planning

- Three-fingered grasps of spherical objects

$$E = \frac{1}{4}(\|X(\alpha_{o3}) - X(\alpha_{o1})\|^2 \|X(\alpha_{o3}) - X(\alpha_{o2})\|^2 - ((X(\alpha_{o3}) - X(\alpha_{o1})) \cdot (X(\alpha_{o3}) - X(\alpha_{o2})))^2)$$



Solution: symmetric grasp

Problem: not general w.r.t. no. of fingers and object geometry

5.5 Grasp Planning

- Max-transfer problem (Ferrari and Canny 92)

$$\vec{\alpha}_o = (\alpha_{o1}^T, \cdots, \alpha_{ok}^T)^T$$

$$g_0(\vec{\alpha}_o) = \min_{\|F_o\|=1} \max_{\substack{Gx = F_o, \\ P(x) \geq 0}} \frac{1}{\|x\|}$$

Problem: Computational difficulties.

4 Min-analytic-center problem:

Analytic center x^*:
(Boyd et. al. 1996)

$$\min_x \log\det P(x)^{-1}$$
$$\text{s.t. } Gx = F$$
$$P(x) > 0$$

Interpretation: the grasping force x which is farthest from the boundary of the friction cone.

$$g(\vec{\alpha}_o) = \max_{F_o^T A F_o = 1} \min_{\substack{Gx = F_o, \\ P(x) > 0}} \log\det P(x)^{-1}$$

A: Task requirement

Interpretation: optimize worst case analytic center.

5.5 Grasp Planning

□ *Simplification for real-time optimization:*

Center of FC: $L = \{\xi t | \xi = (0,0,1,\cdots,0,0,1)^T, t > 0\} (PCWF)$

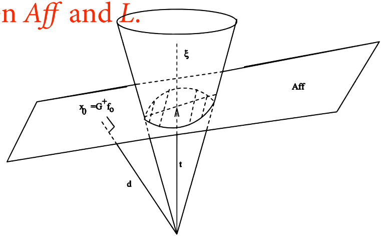
$$L = \{\xi t \big| \xi = (0,0,1,0,\cdots,0,0,1,0)^T, t > 0\} (SFCE)$$

Solution set: $Aff = \{x | Gx = f_o\}$

$x^* \approx$ the intersection point between Aff and L.

$$x^* \approx \frac{\xi}{\sqrt{\xi^T G^T A G \xi}},$$

$$\psi(\vec{\alpha}_o) \approx \log \frac{(\xi^T G^T A G \xi)^k}{\Pi_{i=1}^k \mu_i^2}$$



Problem 4: (simplified)
Find $\vec{\alpha}_o$ s.t. it minimizes $\psi_1(\vec{\alpha}_o) = \xi^T G^T A G \xi$

Note: Optimization of ψ_1 leads to antipodal and symmetric grasps, respectively.

5.5 Grasp Planning

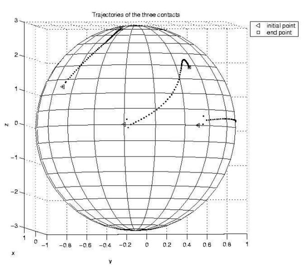
□ *Simulation results:*

◇ *Example: A 3-fingered hand manipulating an ellipsoid*

- Minimize $\psi(\vec{\alpha}_o)$

$$C(\alpha_{o_i}) = \begin{bmatrix} a\cos u_{oi}\cos v_{oi} \\ b\cos u_{oi}\sin v_{oi} \\ c\sin u_{oi} \end{bmatrix}$$

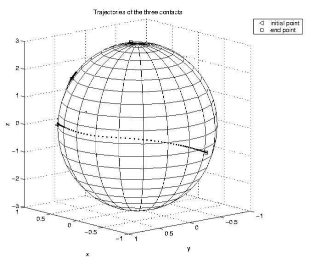
$$c = 3a = 3b = 3$$



5.5 Grasp Planning

- Initial contacts (not force closure)

$$\alpha_{o1} = (0,0)^T$$
$$\alpha_{o2} = (0,\pi/4)^T$$
$$\alpha_{o3} = (\pi/8,-\pi/4)^T$$



Advantages of the quality function approach:

- Objects with arbitrary geometry
- Arbitrary number of fingers
- Ability for real-time contact points servoing

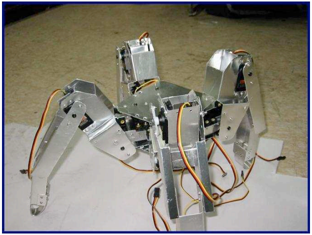
5.6 Grasp Force Optimization

□ *Grasping Force Optimization:*

Problem 5:
Given $F_o \in \mathbb{R}^p$, find $x \in FC$ s.t. $Gx = F_o$ and x minimizes some suitable cost function.

Other Applications

1. Optimal force distribution for multilegged robots;
2. Force control for cable-driven parallel robots



Legged robot

parallel robot

□ *Wrench balance constraint:*

$$Gx = F_o = (f_{o_1}, \dots, f_{o_6})^T \Leftrightarrow \mathrm{Tr}(B_i P(z)) = f_{oi}, i = 1, \dots, 6 \qquad (*)$$

Ω_P is convex (intersection of a convex cone with a convex hyperplane)

Sketch of Proof for $(*)$

$$\mathbb{R}^{2k\times 2k} \mapsto P(x) = \begin{bmatrix} S_1^1 & & \\ & \ddots & \\ & & 0 \end{bmatrix} x_1 + \begin{bmatrix} S_1^2 & & \\ & \ddots & \\ & & 0 \end{bmatrix} x_2$$

$$+ \begin{bmatrix} S_1^3 & & \\ & \ddots & \\ & & 0 \end{bmatrix} x_3 + \cdots + \begin{bmatrix} 0 & 0 \\ & \ddots \\ 0 & S_k^s \end{bmatrix} x_m, m = 3k$$

$$S_1^1 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, S_1^2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, S_1^3 = \begin{bmatrix} \mu_1 & 0 \\ 0 & \mu_1 \end{bmatrix}$$

$$Gx = F_0 \Rightarrow \begin{bmatrix} G_{11} & \cdots & G_{1m} \\ \vdots & \ddots & \vdots \\ G_{61} & \cdots & G_{6m} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_m \end{bmatrix} = \begin{bmatrix} f_{o1} \\ \vdots \\ f_{o6} \end{bmatrix}$$

5.6 Grasp Force Optimization

$$\begin{cases} \mathrm{Tr}(B_1 P(x)) = f_{o1} \\ \vdots \\ \mathrm{Tr}(B_6 P(x)) = f_{o6} \end{cases}, B_1 = \begin{bmatrix} B_1^1 & & 0 \\ & \ddots & \\ 0 & & B_1^k \end{bmatrix} \in \mathbb{R}^{2k\times 2k}$$

$$\begin{cases} \mathrm{Tr}(B_1^1 S_1^1) = G_{11} \\ \mathrm{Tr}(B_1^1 S_1^2) = G_{12} \\ \mathrm{Tr}(B_1^1 S_1^3) = G_{13} \end{cases}, B_1^1 = \begin{bmatrix} b_{11} & b_{12} \\ b_{12} & b_{22} \end{bmatrix} \Rightarrow \begin{cases} b_{11} - b_{22} = G_{11} \\ 2b_{12} = G_{12} \\ b_{11} + b_{22} = \dfrac{G_{13}}{\mu_1} \end{cases}$$

$$\Rightarrow B_1^1 = \begin{bmatrix} \frac{1}{2}(G_{11} + \frac{G_{12}}{\mu_1}) & \frac{G_{12}}{2} \\ \frac{G_{12}}{2} & \frac{1}{2}(\frac{G_{13}}{\mu_1} - G_{11}) \end{bmatrix}$$

The rest of $B_1^i, i = 2, \ldots, k$ and thus $B_j^i, j = 2, \ldots, 6$ can be figured out in a similar manner.

5.6 Grasp Force Optimization

□ *Wrench balance constraint (continued):*

$$\Omega_P = \{x \in \mathbb{R}^n | P(x) > 0, \mathrm{Tr}(B_i P) = f_{o_i}, i = 1, \ldots, 6\}$$

Ω_P is convex (intersection of a convex cone with a convex hyperplane)

Problem 3 (a): Max-det **Problem**

$$\min \Phi(P) = \mathrm{Tr}(CP) + \log\det P^{-1}$$
$$\text{subject to } \mathrm{Tr}(B_i P) = f_{oi}, i = 1, \ldots, 6$$
$$P > 0$$

or

$$\min \Phi(z) = C^T z + \log\det P^{-1}(z)$$
$$\text{subject to } Gx = F_o$$
$$P(x) = S_0 + \sum_{i=1}^m x_i S_i > 0, i = 1, \ldots, m$$
$$c_i = \mathrm{Tr}(CS_i)$$

5.6 Grasp Force Optimization

Configuration space $S_{++}^n = \{P \in \mathbb{R}^{n\times n} | P^T = P, P > 0\}$: Riemannian manifold of dimension $\frac{n(n+1)}{2}$

$T_p S_{++}^n = \{\xi \in \mathbb{R}^{n\times n} | \xi^T = \xi\} = S^n$: $n \times n$ symmetric matrices.

Euclidean metric $\langle , \rangle : T_p S_{++}^n \times T_p S_{++}^n \mapsto \mathbb{R}^n, (\xi, \eta) \mapsto \mathrm{Tr}(\xi\eta)$

◇ *Example:* $S_{++}^2 = \{P \in \mathbb{R}^{2\times 2} | P = P^T, P > 0\}$

$\left\{P = \begin{bmatrix} P_{11} & P_{12} \\ P_{12} & P_{22} \end{bmatrix} \middle| P > 0\right\} \Leftrightarrow P_{11} > 0, P_{11}P_{22} - P_{12}^2 > 0$

$\Rightarrow \{P | P > 0\} \cong \left\{ \begin{bmatrix} P_{11} \\ P_{12} \\ P_{22} \end{bmatrix} \in \mathbb{R}^3 \middle| P_{11} > 0, P_{11}P_{22} - P_{12}^2 > 0 \right\}$: open set in $\mathbb{R}^3$

$T_P S_{++}^2 = \{B \in \mathbb{R}^{2\times 2} | B = B^T\}$: vector space of dimension 3.

$\ll B, C \gg = \mathrm{Tr}(BC) = b_{11}c_{11} + b_{12}c_{12} + b_{12}c_{12} + b_{22}c_{22}$

Dimension of S^n: $\frac{n^2 - n}{2} + n = \frac{n^2 + n}{2}$

5.6 Grasp Force Optimization

◇ *Example:* $S^n = T \oplus T^\perp$

Assumption: $\{B_i\}, i = 1, \ldots, 6$ are linearly independent. By Gram-Schmidt process, orthonormalize the B_i's if necessary.

Thus $\mathrm{Tr}(B_i B_j) = \begin{cases} 1, i = j \\ 0, i \neq j \end{cases}$

Let $T = \{\eta \in S^n | \mathrm{Tr}(B_i \eta) = 0, i = 1, \ldots, 6\}$ be the subspace of constrained velocities, with $\dim(T) = \frac{1}{2} n(n+1) - 6$ ($\dim T_q Q$)

$T^\perp = \mathrm{span}\{B_1, \ldots, B_6\}$ ($T_q Q^\perp$)

Property 6:

- $\Phi(P)$ is a convex function
- Ω_z is a convex set

5.6 Grasp Force Optimization

Let $Q \in S^n_{++}$ be s.t. $P = Q^2$, $P(t) = Qe^{Q^{-1}\xi t Q^{-1}}Q$ satisfies:

$$P(0) = P, \dot{P}(0) = \xi$$

$$\Rightarrow D\Phi(p)(\xi) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \Phi(P(t)) \quad = \mathrm{Tr}(C\xi) - \mathrm{Tr}(P^{-1}\xi)$$

where the second term follows from:

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \log\det P^{-1}(t) = -\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \log\det P(t)$$

$$= -\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} (\log\det Q + \log\det e^{Q^{-1}\xi t Q^{-1}})$$

$$= -\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \log e^{\mathrm{Tr}(Q^{-1}\xi t Q^{-1})}$$

$$= -\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \mathrm{Tr}(Q^{-1}\xi t Q^{-1}) = -\mathrm{Tr}(Q^{-1}\xi Q^{-1}) = -\mathrm{Tr}(P^{-1}\xi)$$

5.6 Grasp Force Optimization

$\Rightarrow \nabla\Phi(P) \in S^n$ is defined by

$$\mathrm{Tr}(\nabla\Phi(P)\xi) = D\Phi(P)(\xi), \forall \xi \in S^n$$

$$\Rightarrow \nabla\Phi(P) = C - P^{-1}$$

$$\Pi : S^n \mapsto T : \nabla\Phi(P) \mapsto \nabla_T\Phi(P)$$

$$\nabla_T\Phi(P) = C - P^{-1} - \sum_{i=1}^{6} \gamma_i B_i,$$

$$\gamma_i = \mathrm{Tr}(B_i(C - P^{-1})), i = 1, \ldots, 6$$

5.6 Grasp Force Optimization

Constraint subspace: $T = \{\eta \in S^n | \mathrm{Tr}(B_i \eta) = 0, j = 1, \ldots, 6\}$

Euclidean gradient: $\nabla_T \Phi(P) = C - P^{-1} - \sum_{i=1}^{6} \gamma_i B_i$

$$\gamma_i = \mathrm{Tr}(B_i(C - P^{-1}))$$

□ *Computation of $D^2\phi(P)$:*

Consider the curve $P(t) = Qe^{Q^{-1}\eta t Q^{-1}}$, $P(0) = Q^2 = \Gamma$, $\dot{P}(0) = \eta$. Then,

$$D^2\phi(\Gamma)(\xi, \eta) = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} D\phi(P(t))(\xi) = \mathrm{Tr}(\Gamma^{-1}\xi\Gamma^{-1}\eta), \forall \xi, \eta \in S^n$$

and $D^2\phi(P)(\xi, \xi) = \mathrm{Tr}(\Gamma^{-1}\xi\Gamma^{-1}\xi) > 0, \forall \xi \neq 0$
$\Rightarrow \phi(\cdot)$ is a convex function. Define $\ll \xi, \eta \gg_g = \mathrm{Tr}(\Gamma^{-1}\xi\Gamma^{-1}\eta)$

5.6 Grasp Force Optimization

Algorithm 1: Dikin-type Euclidean algorithm (BFM98)

$$P_{k+1} = P_k - \alpha_k \frac{\nabla_T \Phi(P_k)}{\|\nabla_T \Phi(P_k)\|_M},$$

$$\|\nabla_T \Phi(P_k)\|_g = \sqrt{Tr(P_k^{-1} \nabla_T \Phi(P_k) P_k^{-1} \nabla_T \Phi(P_k))}$$

$$P_k > 0, \alpha_k \in [0,1) \Rightarrow P_{k+1} > 0$$

Using line-search method for the optimal α_k^*

Algorithm 2: Newton algorithm with estimated step size (HHM02)

$$P_{k+1} = P_k - \alpha_k (D^2\Phi(P_k))^{-1} \nabla_T \Phi(P_k) = P_k - \alpha_k \nabla_T \Phi(P_k)$$

$$\alpha_k = \frac{1 + 2\lambda(P_k) - \sqrt{1 + 4\lambda(P_k)}}{2(\lambda(P_k))^2}, \lambda(P_k) = \sqrt{\mathrm{Tr}(\nabla_T \Phi \nabla T \Phi)}$$

□ ***LMI Model:***

$$P = P_0 + P_1 x_1 + \cdots + P_m x_m \geq 0$$

Elimination of linear constraints $G \cdot x = F_o$

$$x = G^{\dagger} F_o + Vy, y \in \mathbb{R}^{m-6}, GV = 0$$

$$P = \tilde{P}(y) = \tilde{P}_0 + \tilde{P}_1 y_1 + \cdots + \tilde{P}_{m-6} y_{m-6} \geq 0$$

5.6 Grasp Force Optimization

Algorithm 3: Interior point algorithm (HTL00)

$$\min \Psi(y) = C^T y + \log\det \tilde{P}(y)^{-1}$$

$$\text{subject to } \tilde{P}(y) \in \mathbb{R}^{n\times n} = \tilde{P}_0 + \tilde{P}_1 y_1 + \cdots + \tilde{P}_{m-6} y_{m-6} \geq 0$$

$$F(y) = F_0 + F_1 y_1 + \cdots + F_{m-6} y_{m-6} > 0$$

$$c_j = \mathrm{Tr}(C\tilde{P}_j), j = 1, \ldots, m-6$$

Choose F_i s.t. $\mathrm{diag}(\tilde{P}_i, F_i)$'s are linearly independent for $i = 1, \ldots, m-6$ (e.g. $F_0 = 1, F_i = 0, i \geq 1$)

1. Solved efficiently using Interior Point Algorithm
2. Polynomial-type algorithms w.r.t. the problem dimension (i.e. $m-6, n$)

□ *Initial Point Computation:*

Problem 5:
Find an initial point x or y such that $P(x) > 0$ or $\tilde{P}(y) > 0$

[HTL00]

$$\min e^T z = z_{m-6+1} (e = [0 \cdots 0 \;\; 1]^T)$$

subject to $\tilde{P}(z) = 1 \geq 0$

$$F(z) = \tilde{P}_0 + \tilde{P}_1 z_1 + \cdots \tilde{P}_{m-6} z_{m-6} + I z_{m-6+1} \geq 0$$

Solved using the Interior Point Algorithm with initial point $z = [0, \cdots, -\lambda_{\min}(\tilde{P}_0) + \beta]^T, \beta > 0$.

□ *Algorithm analysis & evaluation:*

Property 8: Quadratic convergence property

$$d(P_{k+1}, P^*) \leq d^2(P_k, P^*)$$

algorithms \ platform	No. of iteration	SUN Ultra 60, UNIX
Algorithm 1 (BFM 98)	5	2s/1000 times
Algorithm 2 (HHM 02)	6	3s/1000 times
Algorithm 3 (HTL 00)	2	2s/1000 times

5.7 Coordinated Control

□ *Coordinated Motion Generation:*

Problem 6: Coordinated finger motion generation
Given desired object velocity $V_{po} \in \mathbb{R}^6$, find fingertip velocity $V_{pf_i} \in \mathbb{R}^6$, that satisfies the non-slippage and the force closure constraints.

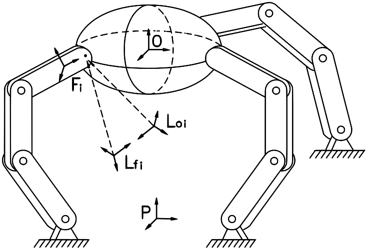
Kinematics

$$g_{po} = g_{pf_i} \cdot g_{f_i lf_i} \cdot g_{l_{f_i} l_{o_i}} \cdot g_{l_{o_i} o}$$

$$V_{po} = \mathrm{Ad}^{-1}_{g_{f_i o}} V_{pf_i} + \mathrm{Ad}^{-1}_{g_{lo_i o}} V_{lf_i lo_i}$$

$$\tilde{V}_{pf_i} = \mathrm{Ad}_{g_{lo_i o}} V_{po} - V_{l_{f_i} l_{o_i}} \qquad (*)$$

$$V_{pf_i} = \mathrm{Ad}_{g_{f_i l_{o_i}}} \tilde{V}_{pf_i}$$



5.7 Coordinated Control

- Grasp optimization

$$\tilde{V}_{pf_i} = \mathrm{Ad}_{g_{l_{o_i}o}} V_{po} - B_{c_i}^c \begin{bmatrix} \omega_x^i \\ \omega_y^i \end{bmatrix}$$

constraints

$$B_{c_i}^c = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}^T$$

- Contact equation

$$\begin{bmatrix} \omega_x^i \\ \omega_y^i \end{bmatrix} = R_{\psi_i}(K_{o_i} + \tilde{K}_{f_i}) M_{o_i} \dot{\alpha}_{o_i}$$

- Grasp quality measure

$$g : \begin{bmatrix} \alpha_{o_1} & \cdots & \alpha_{o_k} \end{bmatrix}^T \in \mathbb{R}^{2k} \to \mathbb{R}$$

- Optimize $F(\cdot)$

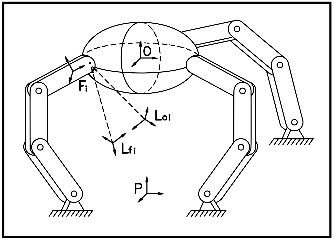
$$\dot{\alpha}_o = -\lambda \nabla g(\alpha_o) = -\lambda \nabla \xi G^T A G \xi, \lambda \in (0,1)$$

5.7 Coordinated Control

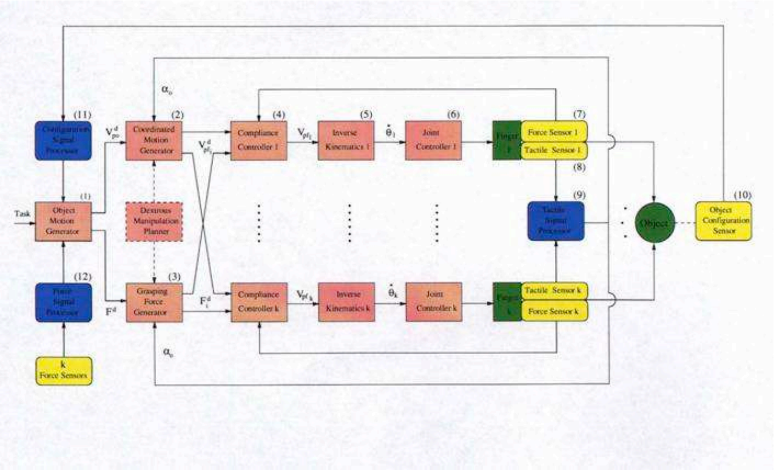
□ *Control System Architecture:*

Problem 7: Formulation of control objectives

1. Desired object velocity V_{po}^d
2. Desired object force f_o^d
3. Suitable grasp quality α_o^d



5.7 Coordinated Control



CoSAM2 – A unified Control System Architecture for Multifingered Manipulation

□ *Coordinated Motion Generation:*

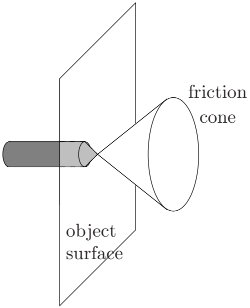
Input	Desired object velocity $V_{po}^d \in \mathbb{R}^6$
Sensors	Tactile sensors
Output	Fingertip velocity $V_{pf_i} \in \mathbb{R}^6$
Constraints	

–Rolling/finger gaiting (non-slippage)
–Force closure

5.7 Coordinated Control

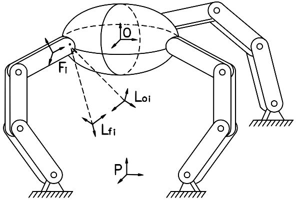
□ *Grasping force generation:*

Input	Desired object force $f_o^d \in \mathbb{R}^6$
Sensors	Tactile and contact force sensors
Output	Fingertip force $x \in \mathbb{R}^m$
Constraints	$-Gx = F^d$
	$-x \in FC$



□ *Compliance Motion Control Module:*

- Input Fingertip velocity $V_{pf_i}^d \in \mathbb{R}^6$ from the CMG module and desired fingertip force F_i^d from the GFG module
- Sensors Contact force sensors
- Output total finger velocity $V_{pf_i} \in \mathbb{R}^6$



$$V_{pf_i} = V_{pf_i}^d + K_{c_i}(F_i^d - F_i^m)$$

$K_{c_i} \in \mathbb{R}^{6\times 6}$: Finger compliance matrix; F_i^m: Measured force.

$$V_{pf_i}^d = \underbrace{\mathrm{Ad}_{g_{f_io}}(\eta_i)V_{po}^d}_{\text{Object motion}} + \underbrace{\mathrm{Ad}_{g_{f_il_{fi}}}(\eta_i)V_{l_{oi}l_{fi}}(\eta_i,\dot{\eta}_i^d)}_{\text{Grasp quality}} + \underbrace{K_{ci}(F_i^d - F_i^m)}_{\text{Object force}}$$

5.7 Coordinated Control

□ *Inverse Kinematics Module:*

- Input Desired Fingertip velocity $V_{pf_i}^d \in \mathbb{R}^6$
- Sensors Finger joint encoders
- Output Finger joint velocity $\dot{\theta} \in \mathbb{R}^{n_i}$
- Constraints

$$V_{pf_i} = J_{f_i}(\theta_i)\dot{\theta}_i, n_i \leq 6$$

–Collision constraints

$$\min_{\dot{\theta}_i \in \mathbb{R}^{n_i}} \langle J_{f_i}\dot{\theta}_i - V_{pf_i}, Q(J_{f_i}\dot{\theta}_i - V_{pf_i}) \rangle$$

subject to $\langle \dot{\theta}_i, M\dot{\theta}_i \rangle \leq \alpha^2$

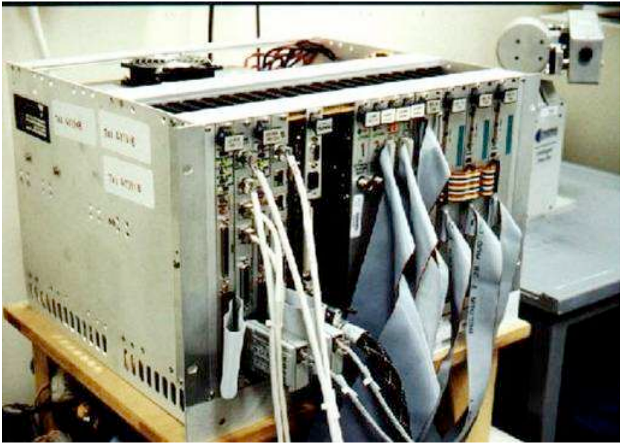
$$\psi(\theta_i) + D\psi(\theta_i)\dot{\theta}_i \leq 0$$

5.7 Coordinated Control

□ *Implementation:*

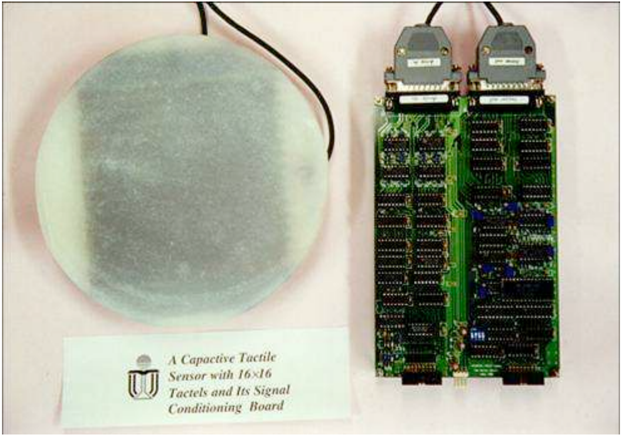
The three-fingered HKUST hand

5.7 Coordinated Control



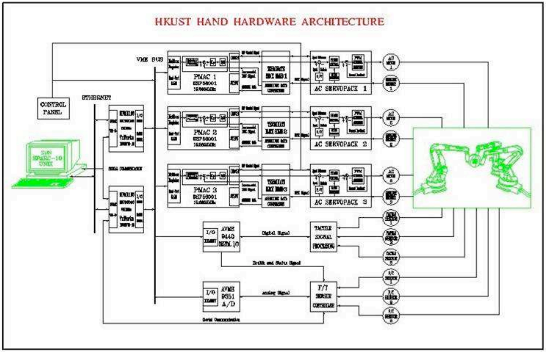
Microprocessor control system for HKUST hand

5.7 Coordinated Control



Tactile sensor and signal conditioning unit for HKUST hand

5.7 Coordinated Control



HKUST hand hardware architecture

5.7 Coordinated Control

HKUST-Hand system

Sensors

Manipulation: single finger and two fingers

Experiment 2
3-Figured Manipulation