Lecture #8

EECS 106B | 206B

NON-HOLONOMIC MOTION PLANNING

Pfaffian Systems

$q \in \mathbb{R}^n$ $\qquad$ $q(\cdot) : [0,T] \to \mathbb{R}^n$

$\dot{q}(t)$ exists

If $\exists$ $w(q)$ a row vector

$[w_1(q) \; . \; . \; . \; w_n(q)]$

$w(q)\,\dot{q} = 0$ is called a Pfaffian constraint

$$w_1(q)\dot{q}_1 + w_2(q)\dot{q}_2 + \dots + w_n(q)\dot{q}_n = 0$$

$$w(q)\,\dot{q} = 0 \quad \Longleftrightarrow \quad h(q) = 0$$

$$h : \mathbb{R}^n \to \mathbb{R}$$

$n-1$

$$\{q : h(q) = 0\}$$

If you know the

$$\Longleftarrow \quad h(q) = 0$$

$$h(q(t)) \equiv 0$$

$$\frac{d}{dt} h(q(t)) = 0$$

$$\frac{d}{dq} h(q) \cdot \dot{q} = 0$$

$$\underbrace{\left[\frac{dh}{dq_1} \; \frac{dh}{dq_2} \; \cdots \; \frac{dh}{dq_n}\right]}_{w(q)} \dot{q} = 0$$

$$w(q)\,\dot{q} = 0$$

$$w_1(q) = \frac{dh}{dq_1} \qquad w_n(q) = \frac{dh}{dq_n}$$

$$w_i(q) = \frac{dh}{dq_i} \qquad w_j(q) = \frac{dh}{dq_j}$$

$$\frac{dw_i}{dq_j} = \underbrace{\frac{d^2h}{dq_j\, dq_i}} \qquad \frac{dw_j}{dq_i} = \underbrace{\frac{d^2h}{dq_i\, dq_j}}$$

$\frac{dw_i}{dq_j} = \frac{dw_j}{dq_i}$ is a necessary condition for

$$w(q)\,\dot{q} = 0 \iff h(q) = 0$$

Analysis

$$w(q)\,\dot{q} = 0 \iff h(q) = 0$$

$$\implies \frac{\partial w_i}{\partial q_j} = \frac{\partial w_j}{\partial q_i}$$

$w(q)\dot{q} = 0$ $\qquad \forall \alpha(q): \mathbb{R}^n \to \mathbb{R}$

$\alpha(q) \neq 0$

$\alpha(q)\, w(q)\dot{q} = 0$

$[\alpha w_1(q) \quad \alpha w_2(q) \quad \cdots \quad \alpha w_n(q)]\,\dot{q} = 0$

? $\exists\ \alpha_i(q) \ni$

$$\frac{d}{dq_j}(\alpha w_i(q)) \neq \frac{d}{dq_i}[\alpha w_j(q)]$$

Theorem If $\exists\ \alpha(q) \ni$

$$\frac{d}{dq_j}(\alpha w_i(q)) = \frac{d}{dq_i}(\alpha w_j(q))$$

Then $\exists\ h(q) = 0$

$w_1^T(q)\dot{q} = 0$ $\qquad w^i(q)\dot{q} = 0$

$$\dot{w}_K^T(q)\,\dot{q} = 0$$

$$w_i^T(q)\,\dot{q} = 0 \quad i = 1, \dots, K$$

$$\overset{?}{\Longrightarrow} \left.\begin{array}{l} h_1(q) = 0 \\ h_2(q) = 0 \\ \vdots \\ h_r(q) = 0 \end{array}\right\}$$

$$r \leq K$$