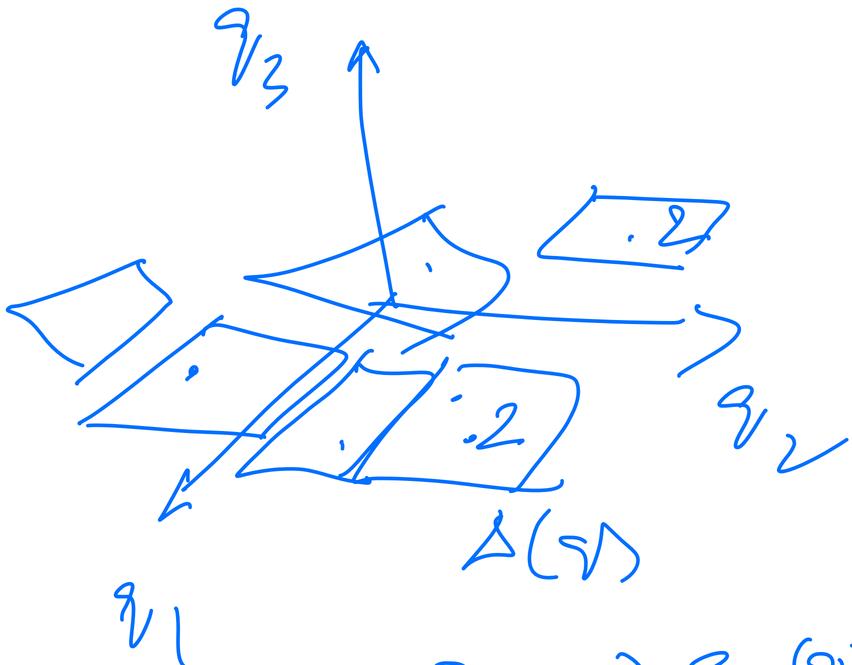
Lecture Notes

Integrability



$$\Delta(q) = \text{span}\{g_1(q), g_2(q)\}$$

Integrability

$\dim M = \dim \Delta$
$= 2$

$\dot{q} = f(q)$

$\Delta(q) = \mathrm{Span}\{f(q)\}$

$\Delta(q) = \mathrm{Span}\{g_1(q), \dots g_p(q)\}$

dimension p

$\mathbb{R}^n$ p dimensional subspace

$\dim M = \dim \Delta(p) = p$

$$M = \left\{ q : \begin{array}{l} h_1(q) = c_1 \\ h_2(q) = c_2 \\ \vdots \\ h_{n-p}(q) = c_{n-p} \end{array} \right\}$$

M INTEGRAL MANIFOLD OF Δ if at each q

$$TM(q) = \Delta(q)$$

Theorem FROBENIUS

$\Delta(q)$ is INTEGRABLE $\Leftrightarrow$ $\Delta(q)$ IS INVOLUTIVE

$$\forall\, g_1(q), g_2(q) \in \Delta \quad [g_1, g_2] \in \Delta$$

$$\Delta(q) = \text{Span}\{g_1(q) \ldots g_m(q)\}$$

$$\dot{q} = g_1(q)u_1 + g_2(q)u_2 + \ldots + g_p(q)u_p$$

If Δ is not involutive

$\bar{\Delta}$ = Involutive Closure of Δ

$$\bar{\Delta} = \{g_1 \ldots g_p, [g_i, g_j], [[g_i, g_j], g_k], \ldots \ldots\}$$

$$\dim \bar{\Delta} = \bar{p}$$

$\bar{\Delta}$ is involutive $\exists$ M of $(n-\bar{p})$ dimension which is the integral manifold of $\bar{\Delta}$

$\alpha_1(q) \ldots \alpha_{n-p}(q)$ such that

- ?

$M = \{ q : \alpha_1(q) = c_1 \ldots \alpha_{n-p}(q) = c_{n-p} \}$

PFAFFIAN CONSTRAINT

$\omega_i(q)\,\dot{q} = 0 \quad i = 1, \ldots, k$

Let $\{ g_1 \ldots g_{n-k} \}$ be the basis of the null space of $\begin{bmatrix} \omega_1(q) \\ \omega_2(q) \\ \vdots \\ \omega_k(q) \end{bmatrix} k$

n

$\dot{q} = g_1(q) u_1 \ldots + g_{n-k}(q) u_{n-k}$

$\Delta = \{ g_1, \ldots g_{n-k} \}$

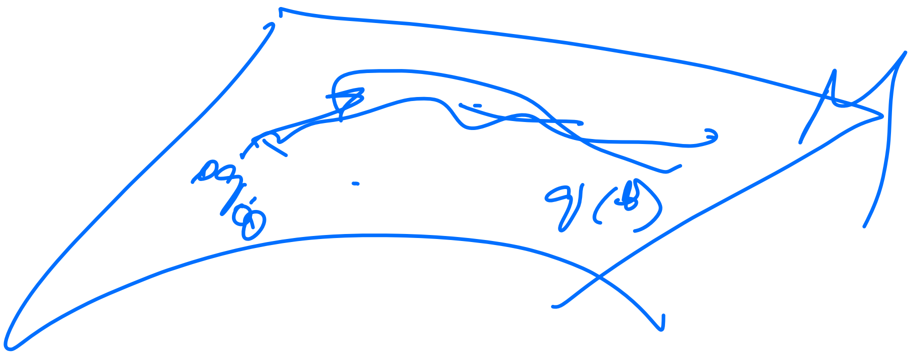
$\bar{\Delta}$: $\dim \bar{\Delta} = p \geq n-k$

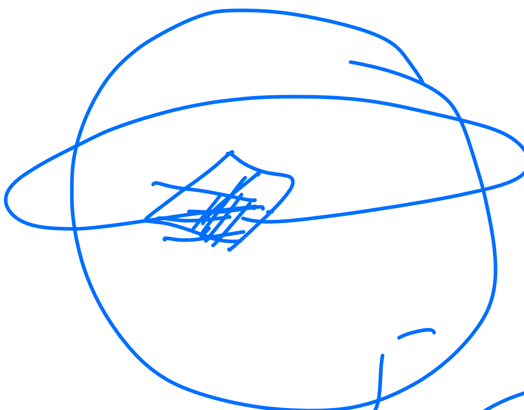
INTEGRAL MAPS

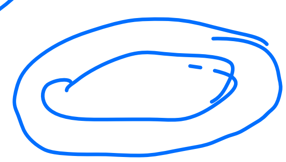
$M \dim n-p$

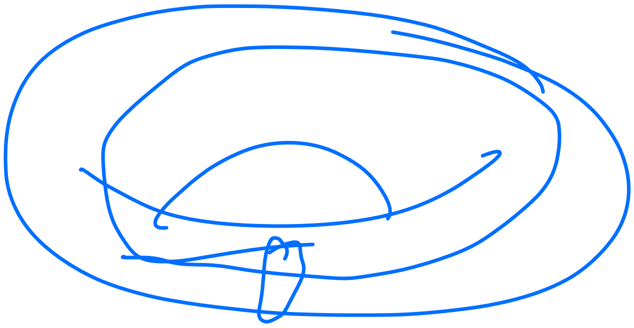
A5

$n-k \le p \le n$









$$SO(2) \quad \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

$$\theta \in [0, 2\pi]$$