Agenda for Today

- NON HOLONOMY
- CONTROLLABILITY & STEERING

FROBENIUS THEOREM
CHOW THEOREM.

ONCE IN A LIFETIME

MLS
PAGE
323-325

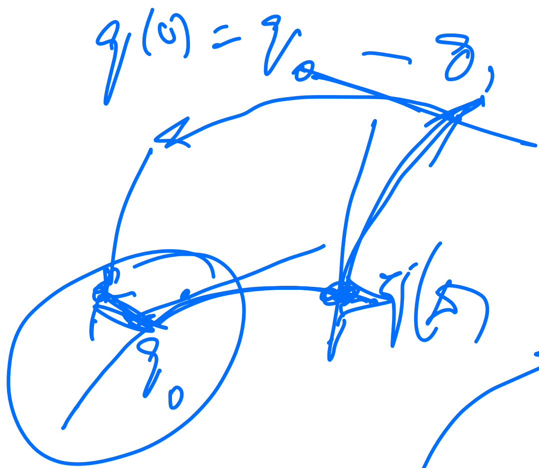
Lie BRACKET

MARIUS SOPHUS
LIE

$$\dot{q} = g_1(q)\, u_1 + g_2(q)\, u_2$$

$$q \in \mathbb{R}^n$$

$g_1(q) \in \mathbb{R}^n \quad g_2(q) \in \mathbb{R}^n$



$u_1 = 1$
$u_2 = 0$

$\dot{q} = g_1(q)$ $\quad q(0) = q_0$

Δ secs.

$q(\Delta)$

$\dot{q} = g_2(q)$ $\quad \left.\begin{matrix} u_1 = 0 \\ u_2 = 1 \end{matrix}\right\} \Delta$

$q(2\Delta)$

$u_1 = -1$
$u_2 = 0$
$\dot{q} = -g_1(q)$
$q(3\Delta)$

$u_1 = 0$
$u_2 = -1$
$\dot{q} = -g_2(q)$

$q(4\Delta)$

$q(4\Delta) - q_0 =$

$$\dot{q} = g_1(q) \qquad q(0) = q_0$$

$$q(t) = q_0 + t\,\dot{q}(t)\Big|_{t=0} + \frac{t^2}{2!}\ddot{q}(t)\Big|_{t=0} + \dots$$

$$\underline{\dot{q} = g_1(q(t))} \qquad \ddot{q} = \frac{d}{dt} g_1(q(t))$$

$$= Dg_1 \cdot \dot{q}$$

$$= [\ \ ]\, g_1(q)$$

$$Dg_1 = \frac{\partial g_1}{\partial q} \in \mathbb{R}^{n \times n}$$

$$\begin{bmatrix} & \vdots & \\ \cdots & \frac{\partial g_{1i}}{\partial q_j} & \cdots \\ & \vdots & \end{bmatrix}$$

$$\ddot{q} = Dg_1\, g_1$$

$$q(t) = q_0 + t\, g_1(q_0) + \frac{t^2}{2} Dg_1\, g_1$$

Call $q(t)$ $x(t)$

$$x(t) = x_0 + t\, g_1(x_0) + \frac{t^2}{2} Dg_1(x_0)\, g_1(x_0) + h.o.t.$$

$$x(\Delta) = x_0 + \Delta\, g_1(x_0) + \underline{\frac{\Delta^2}{2} Dg_1(x_0)\, g_1(x_0)} + O(\Delta^3)$$

$$x(2\Delta) = x(\Delta) + \Delta g_2(x(\Delta)) + \frac{\Delta^2}{2} Dg_2(x(\Delta)) g_2(x(\Delta))$$

$$= x_0 + \Delta g_1(x_0) + \frac{\Delta^2}{2} Dg_1(x_0) g_1(x_0) \quad + O(\Delta^3)$$

$$+ \Delta g_2\left(x(0) + \Delta g_1(x_0) + \frac{\Delta^2}{2} Dg_1(x_0) g_1(x_0)\right) + O(\Delta^3)$$

$$+ \frac{\Delta^2}{2} Dg_2\left(x(0) + \Delta g_1(x_0) + \frac{\Delta^2}{2} Dg_1(x_0) g_1(x_0)\right)$$

$$g_2\left(x(0) + \Delta g_1(x_0) + \frac{\Delta^2}{2} Dg_1(x_0) g_1(x_0)\right)$$

$$+ O(\Delta^3)$$

$$= x_0 + \Delta g_1(x_0) + \frac{\Delta^2}{2} Dg_1(x_0) g_1(x_0) + O(\Delta^3)$$

$$+ \Delta \left\{ g_2(x_0) + \Delta Dg_2(x_0) g_1(x_0) + O(\Delta^2) \right\}$$

$$+ \frac{\Delta^2}{2} \left[Dg_2(x_0) g_2(x_0) + O(\Delta) \right]$$

$$x(2\Delta)$$

$$= x_0 + \Delta\left(g_1(x_0) + g_2(x_0)\right)$$

$$+ \frac{\Delta^2}{2} \boxed{Dg_1(x_0)\, g_1(x_0)} + \Delta^2 Dg_2(x_0)\, g_1(x_0)$$

$$+ \frac{\Delta^2}{2} \boxed{Dg_2(x_0)\, g_2(x_0)} + O(\Delta^3)$$

$$x(3\Delta) = x(2\Delta) - \Delta g_1(x) + - - -$$

$$x(3\Delta) = x_0 + \Delta\left(g_2(x_0)\right)$$

$$+ \frac{\Delta^2}{2}\Big[Dg_2(x_0)\, g_2(x_0)$$

$$+ 2\, Dg_2(x_0)\, g_1(x_0)$$

$$- 2\, Dg_1(x_0)\, g_2(x_0)\Big]$$

$$x(4\Delta) = x_0 + \text{No term of } O(\Delta)$$

$$+ \Delta^2\Big[Dg_2(x_0)\, g_1(x_0)$$

$$- Dg_1(x_0)\, g_2(x_0)\Big]$$

$$- Dg_1(x_0)g_2(x_0)\Big]$$

$$+ O(\Delta^3)$$

$$x(4\Delta) = x_0 + \Delta^2 \Big\{ Dg_2(x_0)g_1(x_0) - Dg_1(x_0)g_2(x_0) \Big\}$$

LIE BRACKET

$$[g_1, g_2] = Dg_2(x)g_1(x) - Dg_1(x)g_2(x)$$

Δ Small

$$x(2\Delta) = x_0 + \Delta g_1(x) + \Delta\{ g_2(x(\Delta)) \}$$

$$g_2\big(x(0) + \Delta g_1(x)\big)$$

$$\Delta\{ g_2(x_0) + Dg_2(x_0)\Delta g_1(x_0) \}$$

$$[g_1, g_2]$$

CHOW

$$\dot{x} = g_1(x)u_1 + g_2(x)u_2$$

$$[g_1, g_2]$$

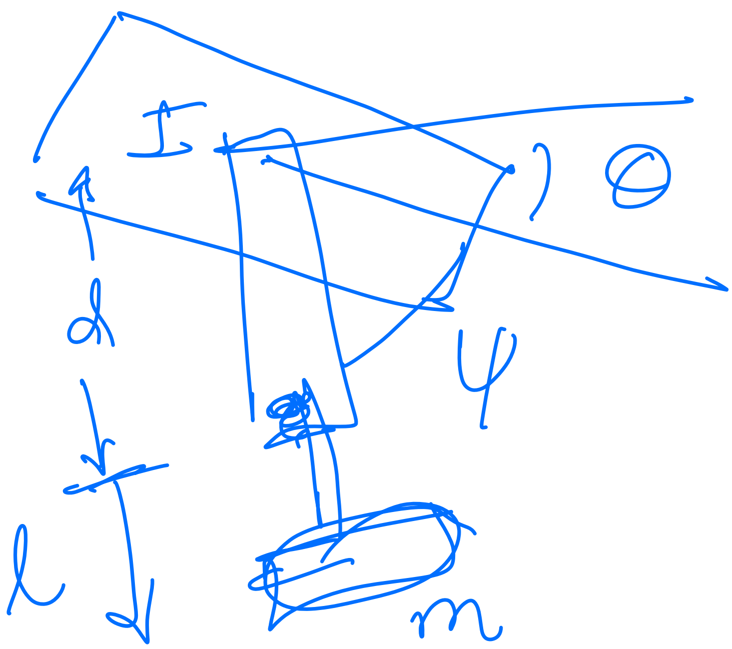
$$[g_2, [g_1, g_2]]$$

$$\{ [g_1, g_2, [g_1, g_2], [g_1, [g_1, g_2]], [g_2, [g_1, g_2]] \}$$

CONTROLLABILITY LIE ALGEBRA

At each x $\Big|$ $\mathrm{Span}\{ g_1(x), g_2(x), [g_1, g_2](x), \ldots \} = \mathbb{R}^n$

$\Rightarrow$ COMPLETELY CONTROLLABLE



$$I\ddot{\theta} + m(l+d)(\ddot{\theta}+\ddot{\varphi}) = 0$$

$$\begin{bmatrix} m(l+d)^2 & 0 & I+m(l+d)^2 \end{bmatrix} \begin{bmatrix} \ddot{\varphi} \\ \ddot{l} \\ \ddot{\theta} \end{bmatrix} = 0$$

$$w_1(x) \qquad x = \begin{bmatrix} \varphi \\ \\ \\ \end{bmatrix}$$

$$w_1(x) \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = 0$$

$$w_1(x) \begin{bmatrix} 1 \\ 0 \\ \dfrac{-m(l+d)^2}{I+m(l+d)^2} \end{bmatrix} = 0$$

$w_1(x) = 0$

$$\Longleftrightarrow \quad \dot{x} = \begin{bmatrix} 1 \\ 0 \\ \dfrac{-m(l+d)^2}{I+m(l+d)^2} \end{bmatrix} u_1 + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} u_2$$

$l = x_2$

$$\dot{x} = \underbrace{\begin{bmatrix} 1 \\ 0 \\ \dfrac{m(x_2+d)^2}{I+m(x_2+d)^2} \end{bmatrix}}_{g_1(x)} u_1 + \underbrace{\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}}_{g_2(x)} u_2$$

Span $\{g_1(x), g_2(x)\}$ is 2 dimensional

$$[g_1, g_2] = Dg_2\, g_1 - Dg_1\, g_2$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} g_1$$

$$= -\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & \frac{2\operatorname{Im}(x_2+d)}{(1+\operatorname{Im}(x_2+d))^2} & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$= -\begin{bmatrix} 0 \\ 0 \\ \end{bmatrix}$$

$$\frac{2 I_m (x_2+d)}{\left(I + m(x_2+d)^2\right)^2}$$

$$g_1 = \begin{bmatrix} 1 \\ 0 \\ \dfrac{m(x_2+d)^2}{I+(x_2+d)^2} \end{bmatrix} \qquad g_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

Rank $\left\{ g_1, g_2 \quad [g_1, g_2] \right\} = 3$

$$[g_1(x), g_2(x)] = Dg_2(x)\,g_1(x) - Dg_1(x)\,g_2(x)$$

$$g_1(x) = Ax$$

$$g_2(x) = Bx$$

$$[g_1(x), g_2(x)] = (BA - AB)x$$

$$= (BA - AB)x$$

$$[g_1, g_2] = -[g_2, g_1]$$

$$\omega_1 \times \omega_2 \doteq \hat{\omega}_1 \, \omega_2$$