EECS 106B | 206B

LECTURE 5 1/31/2023

Chapter 8 of Sastry 1999

S.I.S.O.

$$\dot{x} = f(x) + g(x)u \qquad x \in \mathbb{R}^n$$

$$y = h(x) \qquad u \in \mathbb{R}$$

$$y \in \mathbb{R}$$

$f(x) : \mathbb{R}^n \to \mathbb{R}^n$

$g(x) : \mathbb{R}^n \to \mathbb{R}^n.$

If $f(x) = 0$ $\qquad \dot{x} = g(x)u$

DRIFT FREE

$u = 0$

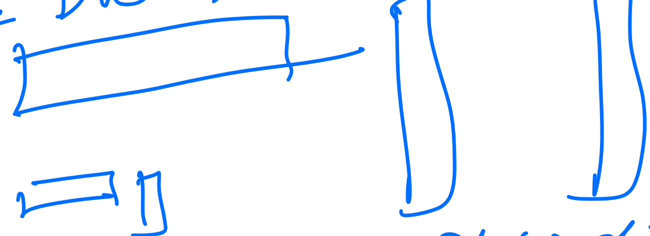
$\dot{x} = f(x).$

I/O FEEDBACK LINEARIZATION

$$\dot{y} = \frac{d}{dt} h(x)$$

$$= Dh(x)\, \dot{x}$$

$$= Dh(x)\, [f(x) + g(x) u]$$



$$= Dh(x) f(x) + Dh(x) g(x) u$$

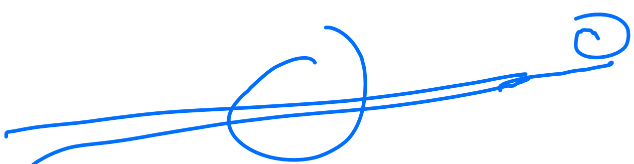
$$\dot{y} = \underbrace{L_f h(x)}_{\text{Lie derivative of } h \text{ in the direction of } f} + L_g h(x) u$$

If $L_g h(x) \neq 0$ W.L.O.G $x_0 = 0$

$f(x_0) = 0$

U nbd of x_0

$L_g h(x) \neq 0 \quad x \in U$

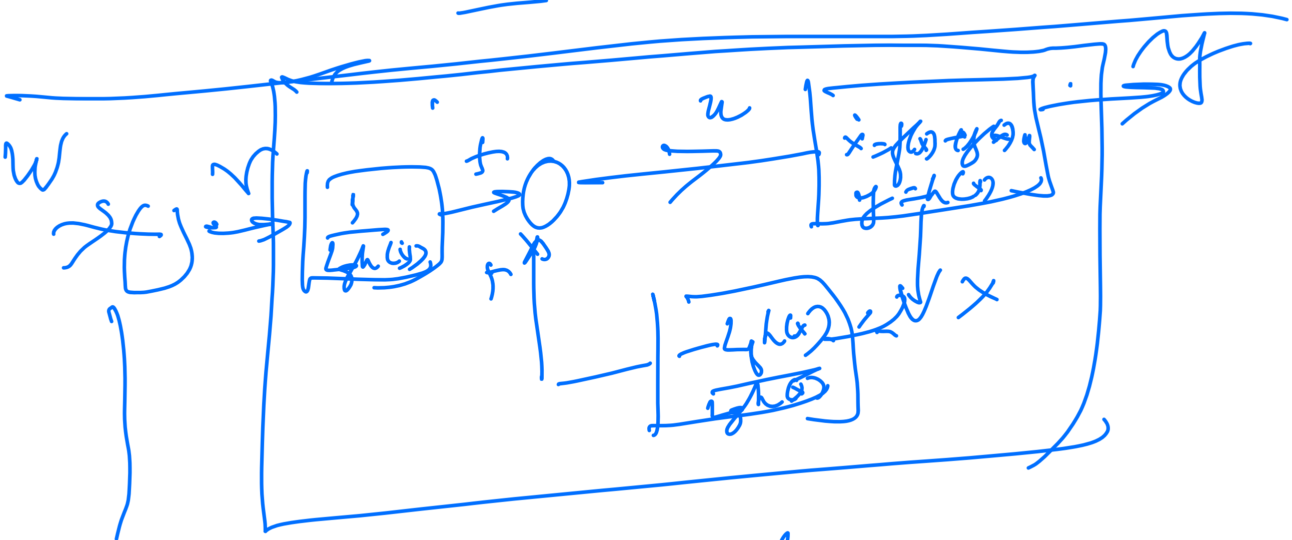


$L_g h(x) \neq 0$

$$u = \frac{1}{L_g h(x)} \left[-L_f h(x) + v \right]$$

$$\dot{y} = \cancel{L_f h(x)} + \frac{\cancel{L_g h(x)}}{\cancel{L_g h(x)}} \left[\cancel{-L_f h(x)} + v \right]$$

$$= v$$



Wants $y(\cdot)$ to track $y_{des}(\cdot)$

$$v \rightarrow \boxed{1/s} \rightarrow y$$

$$\dot{y} = v = \dot{y}_{des}(t) + \alpha(y_{des}(t) - y(t))$$

$$y(t) - y_{des}(t) = e(t)$$

$$\dot{e} + \alpha e = 0$$

$$v = \ddot{y}_{des}(t) + \alpha(\dot{y}_{des}(t) - \dot{y}(t))$$

If $L_g h(x) = 0$

$$\dot{y} = L_f h(x)$$

$$\ddot{y} = \frac{d}{dt}\left(L_f h(x)\right)$$

$$= D L_f h(x) \cdot \dot{x}$$

$$= D L_f h(x)\left[f(x) + g(x)u\right]$$

COMP. OF $L_g L_f$

$$= L_f \circ L_f h(x) + L_g \circ L_f h(x) u$$

$L_g(L_f h(x))$

$L_f^2 h(x)$

$$!= L_f^2 h(x) + L_g L_f h(x) u$$

$$\ddot{y} = L_f^2 h(x) + L_g L_f h(x) u$$

$$L_g L_f h(x) \neq 0 \quad x \in U$$

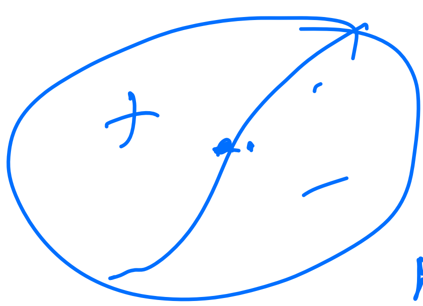
$$u = \frac{1}{L_g L_f h(x)} \left[-L_f^{2} h(x) + v \right]$$

TRACKING $\quad \ddot{y} = v \qquad v \to \boxed{1/s^2} \to y$

$y(\cdot) \quad y_{des}(\cdot)$

$$v = \ddot{y}_{des} + \alpha_1 (\dot{y}_{des} - \dot{y}) + \alpha_2 (y_{des} - y)$$

$L_f h(x)$



$L_g h(x) = 0$

NOT REGULAR

DEFN

NL $\begin{cases} \dot{x} = f(x) + g(x) u \\ y = h(x) \end{cases} \qquad f(x_0) = 0, \quad x \in U$

NL control system is said to have <u>strict</u> relative degree r

if

$$\begin{cases} L_g h(x) = L_g L_f h(x) = \dots = L_g L_f^{r-2} h(x) = 0 \\ L_g L_f^{r-1} h(x) \neq 0 \end{cases}$$

$$\begin{cases} \dot{y} = L_f h(x) \\ \ddot{y} = L_f^2 h(x) \\ \vdots \\ y^{(r)} = L_f^r h(x) + L_g L_f^{r-1} h(x) u \end{cases}$$

$$u = \frac{1}{L_g L_f^{r-1} h(x)} \left[-L_f^r h(x) + v \right]$$

$$y^{(r)} = v$$

$L_f^{r-1} h(x)$ $\quad$ $L_f^{r-2} h(x)$

$$v = y_{des}^{(r)}(t) + \alpha_1 \left[y_{des}^{r-1}(t) - y^{r-1}(t) \right] + \alpha_2 \left[y_{des}^{r-2}(t) - y^{(r-2)}(t) \right] + \dots \alpha_n \left[y_{des}^{(1)} - y^{(1)} \right]$$

$h(x)$

$$e^{(r)} + \alpha_1 e^{r-1} + \dots + \alpha_n e = 0$$

Zeros $\quad s^r + \alpha_1 s^{r-1} + \dots + \alpha_n = 0 \subset \mathring{\mathbb{C}}_-$

$$\bar{e} = \begin{bmatrix} e \\ \dot{e} \\ \vdots \end{bmatrix}$$

$$\begin{bmatrix} \vdots \\ \dot e^{n-1} \end{bmatrix}$$

$$\dot{\bar e} = \begin{bmatrix} 0 & 1 & & 0 \\ & 0 & \ddots & \\ & & & 1 \\ \alpha_n & \cdots & & \alpha_1 \end{bmatrix} \bar e$$

$$\sigma(A) \subset \bar{\mathbb{C}}^-$$

If NL has rel. degree r

$$u = \frac{1}{L_g L_f^{r-1} h(x)} \left[-L_f^{r} h(x) + v \right]$$

Where is the nonlinearity?

NORMAL FORM

Define $\xi_1 := h(x)$

$\xi_2 = L_f h(x)$

$\vdots$

$\xi_r = L_f^{r-1} h(x)$

$\eta_1(x)$

$\vdots$

$\eta_{n-r}(x)$

$\begin{bmatrix} \xi_1 \\ \vdots \\ \xi_r \end{bmatrix} \in \mathbb{R}^r$ INDEPENDENT

$\begin{bmatrix} D\xi_1 \\ \vdots \\ D\xi_r \end{bmatrix} \in \mathbb{R}^{r \times n}$, $x \in U$, Rank r

Chosen to be indep of $\xi_1, \dots, \xi_r$ and to complete the basis

$$\begin{bmatrix} \xi_1 \\ \vdots \\ \xi_r \\ \eta_1 \\ \vdots \end{bmatrix} \xleftarrow{\Phi} x$$

$x \to [\xi$

η_{n-r} ... x ... (ξ, η)

1. C.B.S.T. $\Phi : \mathbb{R}^n \to \mathbb{R}$ is invertible in U

$\Phi^{-1} : \begin{bmatrix} \xi \\ \eta \end{bmatrix} \to x$

$\xi_1 = h(x)$ $\quad \dot{\xi}_1 = \xi_2$

$\xi_2 = L_f h(x)$ $\quad \dot{\xi}_2 = \xi_3$

$\dot{\xi}_r = \underline{L_f^r h(x)} + \underline{L_g L_f^{r-1} h(x)} u$

$\dot{\eta}_1 = \underline{L_f \eta_1} + L_g \eta_1 u$

$\dot{\eta}_2 = L_f \eta_2 + L_g \eta_2 u$

$\dot{\eta}_{n-r} = L_f \eta_{n-r} + L_g \eta_{n-r} \cdot u$

$\dot{\xi}_1 = \xi_2$

$\dot{\xi}_2 = \xi_3$

$\dot{\xi}_r = \underline{L_f^r h\left(\Phi^{-1}\begin{pmatrix}\xi \\ \eta\end{pmatrix}\right)} + \underline{L_g L_f^{r-1} h\left(\Phi^{-1}\begin{pmatrix}\xi \\ \eta\end{pmatrix}\right)} u$

$\dot{\eta}_1 = L_f \eta_1\left(\Phi^{-1}\begin{pmatrix}\xi \\ \eta\end{pmatrix}\right) + L_g \eta_1\left(\Phi^{-1}\begin{pmatrix}\xi \\ \eta\end{pmatrix}\right) u$

$\dot{\eta}_{n-r} = L_f \eta_{n-r}\left(\Phi^{-1}\begin{pmatrix}\xi \\ \eta\end{pmatrix}\right) + L_g \eta_{n-r}\left(\Phi^{-1}\begin{pmatrix}\xi \\ \eta\end{pmatrix}\right) u$

$\dot{\xi}_1 = \xi_2$ $\qquad L_g L_f^{r-1} h(x)$

$L_f^{r-1} h(x)$

$\dot{\xi}_{r-1} = \xi_r$

$\dot{\xi}_r = b(\xi,\eta) + a(\xi,\eta)u$

NORMAL FORM

$\dot{\eta}_1 = p_1(\xi,\eta) + q_1(\xi,\eta)u$

$\vdots$

$\dot{\eta}_{n-r} = p_{n-r}(\xi,\eta) + q_{n-r}(\xi,\eta)u$

$y = \xi_1$

$$u = \frac{1}{a(\xi,\eta)}\left[-b(\xi,\eta) + v\right]$$

$\dot{\xi}_1 = \xi_2$

$\dot{\xi}_2 = \xi_3$

$\vdots$

$\dot{\xi}_r = v$

$\dot{\eta}_1 = p_1(\xi,\eta) + q_1(\xi,\eta)u$

$\vdots$

$\dot{\eta}_{n-r} = p_{n-r}(\xi,\eta) + q_{n-r}(\xi,\eta)u$

$y = \xi_1$

UNOBSERVABLE

If $r = n$ FULL STATE LINEARIZATION

$$n \left\{ \begin{array}{l} \dot{\xi}_1 = \xi_2 \\ \vdots \\ \dot{\xi}_n = a(\xi) + b(\xi)u \end{array} \right.$$

$$y = \xi_1$$

$$u = \frac{1}{b(\xi)}\left[-a(\xi) + v\right]$$

$$y^{(n)} = v$$

$$\hat{h}(s) = \frac{\hat{n}(s)}{\hat{d}(s)} \in \mathbb{R}(s)$$

$\hat{n}(s), \hat{d}(s) \in \mathbb{R}[s]$

$\text{degree } \hat{d}(s) - \text{degree } \hat{n}(s) = \text{rel. degree of } \hat{h}(s)$

$$\boxed{\begin{array}{l} \dot{x} = Ax + bu \\ y = cx \end{array}}$$

$A \in \mathbb{R}^{n \times n}$

$b \in \mathbb{R}^n$

$c^T \in \mathbb{R}^n$

$$\dot{y} = c\dot{x}$$

$$= cAx + \underbrace{cbu}_{} \qquad cb = 0$$

$$= cAx$$

$$y^{(n)} = cA^{2}x + cA^{s-1}bu$$

$$(-A^{s}x + v)$$

$u = \frac{1}{cA^{-1}b}\left(-cA^{-1} \cdots\right)$

State Feed back law

$\hat{y} = (v)$

$\frac{1}{s}$

$c(sI-A)^{-1}b = \frac{\hat{n}(s)}{\hat{d}(s)}$ $\qquad \hat{n}(s)\, s^2$

$(s-\lambda_1)\cdots(s-\lambda_n)$

Cancelling Zeros & putting residual poles at $s = \infty$

Zeros in $\mathbb{C}^-$ MIN. PHASE

$\dot{x} = Ax + bu$

$s\hat{x}(s) - x(0) = A\hat{x}(s) + b\hat{u}(s)$

$\hat{y} = c\,\hat{x}(s)$

$\hat{x}(s) = (sI-A)^{-1}x(0) + (sI-A)^{-1}b\hat{u}(s)$

$\hat{y} = c(sI-A)^{-1}x_0 + \underbrace{c(sI-A)^{-1}b}_{\frac{\hat{n}(s)}{\hat{d}(s)}}\hat{u}(s)$

MINIMUM PHASE

$$\dot{x} = f(x) + g(x)u \qquad x_0 = 0$$

(NL)

$$y = h(x) \qquad y_0 = h(x_0) = 0$$

(NL) with strict rel. degree r is said to be <u>strict</u> minimum phase if

$$\dot{\xi}_1 = \xi_2$$

$$\dot{\xi}_r = b(\xi,\eta) + a(\xi,\eta)u$$

$$\dot{\eta}_1 = q_1(\xi,\eta) + p_1(\xi,\eta)u$$

$$\dot{\eta}_{n-r} = q_{n-r}(\xi,\eta) + p_{n-r}(\xi,\eta)u$$

$$u = \frac{1}{a(\xi,\eta)}\left[-b(\xi,\eta) + \cancel{v}\right] \quad 0$$

$$\left.\begin{aligned} \dot{\xi}_1 &= \xi_2 \\ \dot{\xi}_r &= 0 \end{aligned}\right\} \quad \xi(0) = 0$$

$$\left\{\begin{aligned} \dot{\eta}_1 &= q_1(0,\eta) + p_1(0,\eta)\,u(0,\eta) \\ \dot{\eta}_2 &= q_2(0,\eta) + p_2(0,\eta)\,u_2(0,\eta) \end{aligned}\right.$$

$n-r$

ZERO DYNAMICS OF NL SYSTEM

$$\dot{\eta} = \tilde{q}(\eta) \qquad \eta \in \mathbb{R}$$

$\eta = 0$ is an equil pt

0 is stable $\Rightarrow$ NL is min phase

0 is asy stable $\Rightarrow$ NL is asy. min phase

0 is exp stable $\Rightarrow$ NL is exp mini. phase