EECS 106B / 206B

Robotics

Tu We 11–12 in 333B CORY

or by appointment

Lyapunov Analysis & Stability

M.L.S. Chapter 4: Sec 4.1 & 4.2

$$\dot{x} = f(x, t) \quad x \in \mathbb{R}$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \end{bmatrix} = \begin{bmatrix} f_1(x, t) \\ f_2(x, t) \\ \vdots (x, t) \end{bmatrix}$$

$\begin{bmatrix} \vdots \\ \dot{x}_n \end{bmatrix} \quad f_n(\cdot) \Big]$

$x_e \in \mathbb{R}^n$ is called an equilibrium point if

$f(x_e, t) \equiv 0 \quad \forall t$

$x(0) = x_e \Longrightarrow x(t) \equiv x_e$

Stability is L. ORRERY

LYAPUNOV

N-BODY

3-BODY

PONCARE

$V(x,t)$ WEIERSTRASS

$V(x,t) : \mathbb{R}^n \to \mathbb{R}$

"energy like"

$$\frac{d}{dt} V(x(t),t) = \frac{\partial V}{\partial x}\dot{x} + \frac{\partial V}{\partial t}$$

$$= D_1 V(x,t)\dot{x} + D_2 V(x,t)$$

$$\boxed{}$$

$$= D_1 V(x,t) f(x,t) + D_2 V(x,t)$$

Candidate

$$V(x,t) = KE + P.E.$$

$$\dot{x} = f(x,t)$$

Theorem (Lyapunov) 1892

Assume $V(x,t)$ is an "energy like" function, x_e is an equilibrium point then → (W.L.O.G. $x_e = 0$)

Conc

$\dot{x} = f($

		$V(x,t)$	$\dot V(x,t)$	
①		p.d.f.	≤ 0	x_e is stable
②		p.d.f	$-\dot V(x,t)$ p.d.f	asymp. stable

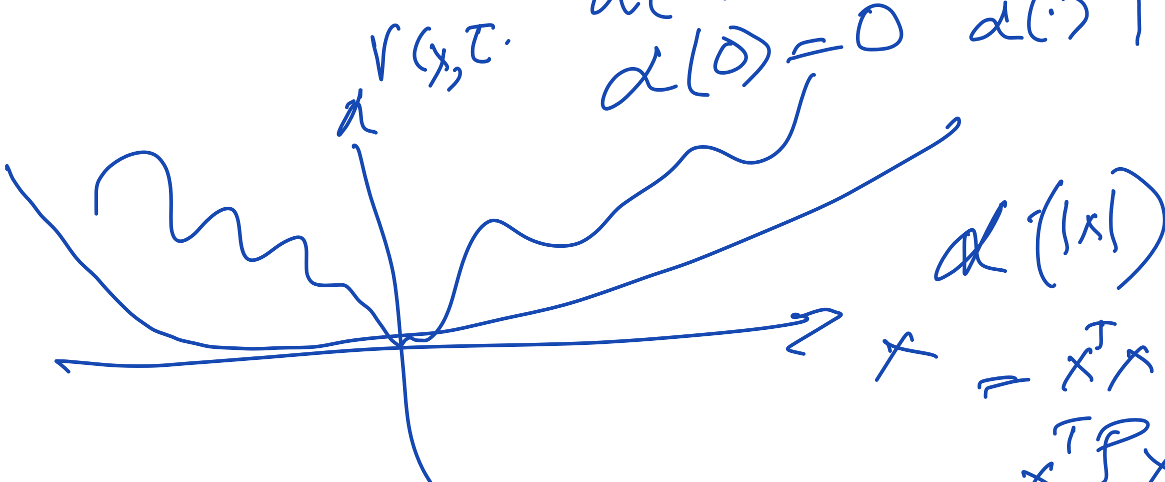
See Table 4.1 of ILS

$V(x,t)$ is p.d.f. or an energy like function of

$$V(x,t) \geq \alpha(|x|)$$

$\alpha(\cdot): R \to \mathbb{R}$

$\alpha(0) = 0$ $\quad \alpha(\cdot) \uparrow$



$x = x^T x$

$x^T P x$

If you can find some

$$\left.\begin{array}{ll} V(x,t) & \text{p.d.f.} \\ -\dot{V} & \text{p.d.f.} \end{array}\right\} \Longrightarrow x_e \text{ is asymp. stable}$$

KHARKIV

FRENCH

TOULON

R. KALMAN
60's

CIXIN LIU

$x \in \mathbb{R}^2 \quad V(x) = x_1^2 + x_2^2$

2 2

$= x_1 + \frac{x_2}{4}$

$x^T P x$ $\quad P = P^T$

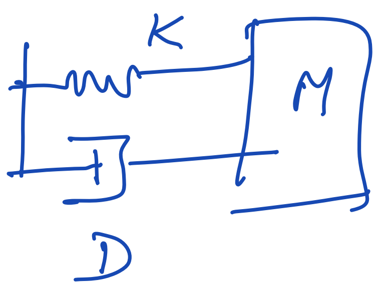
symmetric
pos. def.

$[x_1, x_2]\begin{bmatrix}1 & \\ & 1/4\end{bmatrix}\begin{bmatrix}x_1 \\ x_2\end{bmatrix}$

$= x_1^2 + x_2^4$

$V(x) \quad \sin^2 x_1 + x_2^2$

close to $x_1 = 0$
$x_2 = 0$



$M\ddot{x} + D\dot{x} + Kx = 0$

$x_1 = x \quad x_2 = \dot{x}$

$x = 0$

$$\begin{bmatrix}\dot{x}_1 \\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}0 & 1 \\ -D/M & -K/M\end{bmatrix}\begin{bmatrix}x_1 \\ x_2\end{bmatrix}$$

$D, K, M > 0$ ev.

$$\frac{-D \pm \sqrt{D^2 - 4KM}}{2M}$$

$$V(x_1, x_2) = \tfrac{1}{2}\left(M x_2^2 + K x_1^2\right)$$

$$\dot{V}(x_1, x_2) = \frac{2M x_2 \dot{x}_2 + 2K x_1 \dot{x}_1}{2}$$

$$= -D x_2^2 \leq 0$$

$$V(x_1, x_2) = \frac{1}{2}\begin{bmatrix} x_1 & x_2 \end{bmatrix}\begin{bmatrix} K & \cancel{0} \; -\varepsilon M \\ \cancel{0} \; -\varepsilon M & M \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$P > 0$

$$\dot{V}(x_1, x_2) = -\begin{bmatrix} x_1 & x_2 \end{bmatrix}\begin{bmatrix} \varepsilon K & \tfrac{1}{2}\varepsilon D \\ \tfrac{1}{2}\varepsilon \cancel{M}D & D - \varepsilon M \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$-\dot{V}$ is a p.d.f.

$\boxed{1992}$

FEEDBACK LINEARIZATION

$\dot{x} = f(x)$

$z = \phi(x) : \mathbb{R}^n \to \mathbb{R}^n$

$\dot{z} = D\phi(x)\ \dot{x}$

$= D\phi(x)\ f(x)$

$= D\phi(\phi^{-1}(z)) \cdot f(\phi^{-1}(z))$

$\overset{?}{=} Az$

$\dot{x} = -x$

$z = \tan x$

$\dot{z} = \sec^2 x\ \dot{x}$

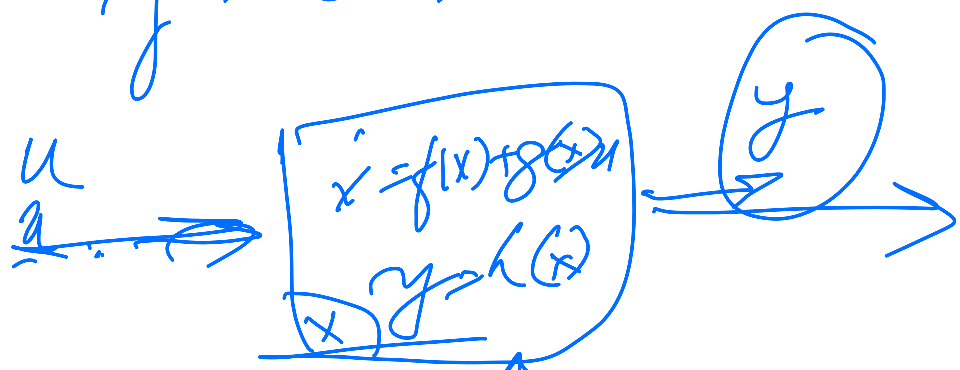
$\dot{z} = -(1+z^2)\tan^{-1} z$

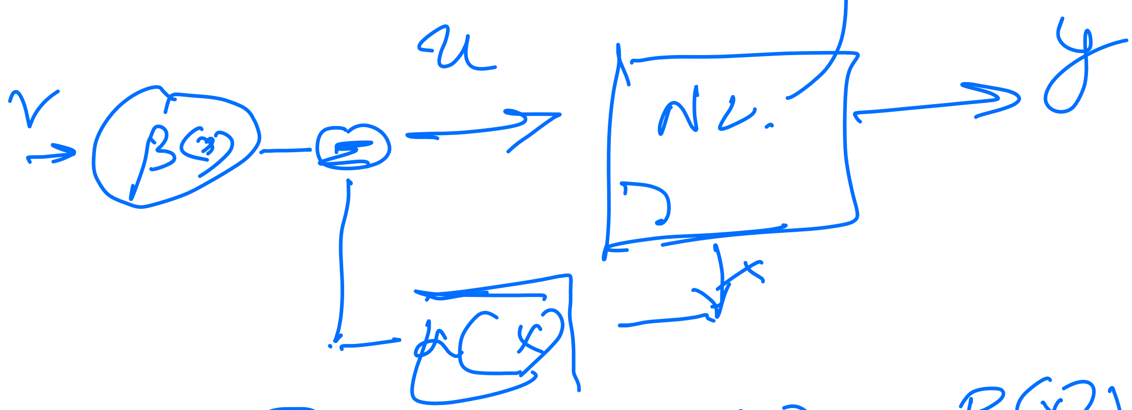
10 INCAGE

Hesat, Su & Meyer (linearization)

$$\dot{x} = f(x) + g(x)u \qquad u \in \mathbb{R}$$

$$y = h(x)$$





$$? \ \exists \quad u = \alpha(x) + \beta(x) v$$

$$\dot{y} = \frac{d}{dt} h(x)$$

$$= \frac{d h(x)}{dx} \cdot \left[f(x) + g(x) u \right]$$

$$= \frac{dh}{dx} f(x) + \frac{dh}{dx} g(x) u$$

$$= L_f h(x) + L_g h(x) u$$

Lie derivative

$$\dot{y} = L_f h(x) + L_g h(x) u$$

If $L_g h(x) \neq 0$ consider

$$u = \frac{1}{L_g h(x)} \left[-L_f h(x) + v \right]$$

$$\alpha(x) + \beta(x) v$$

$$=$$

$$\alpha(x) = \frac{-L_f h(x)}{L_g h(x)} \quad \beta(x) = \frac{1}{L_g h(x)}$$

$$\dot{y} = L_f h(x) + L_g h(x) \left[\frac{-L_f h(x) + v}{L_g h(x)} \right]$$

$$= \cancel{L_f h(x)} - \cancel{L_f h(x)} + v$$

1^{st} ORDER

$$\boxed{\dot{y} = v}$$

WANT

$y(\cdot)$ to track $y_{des}(\cdot)$

$$v = \dot{y}_{des}(t) + \alpha(y_{des}(t) - y(t))$$

$$\dot{y} = \dot{y}_{des}(t) + \alpha(y_{des}(t) - y(t))$$

$$y(t) - y_{des}(t) = e(t)$$

$$\dot{e} + \alpha e = 0$$

$$u = \frac{1}{L_g h(x)} \left[-L_f h(x) + \dot{y}_{des}(t) + \alpha\left(y_{des}(t) - y(t)\right) \right]$$

→ CONTROLS NL

If $L_g h(x) = 0$

$$\dot{y} = L_f h(x)$$

$$\ddot{y} = \frac{d}{dt}\left(L_f h(x)\right)$$

$$= \frac{d}{dx}\left(L_f h(x)\right) \dot{x}$$

$$= \underbrace{\frac{d}{dx}\left(L_f h(x)\right)}\left[f(x) + g(x)u\right]$$

$$= \left[\frac{\partial}{\partial x_1} \frac{\partial}{\partial x_2} \cdots \frac{\partial}{\partial x_n}\right]\left(\left[\;\right] + \left[\;\right]u\right)$$

$$= L_f(L_f h) + L_g(L_f h)u$$

$$= L_f^2 h(x) + \underline{L_g L_f h(x)\, u}$$

$$(L_g \circ L_f h)$$

$L_g h(x) = 0$ If $L_g L_f h(x) \neq 0$

$$u = \frac{1}{L_g L_f h(x)}\left[-L_f^2 h(x) + v\right]$$

$$\ddot{y} = v$$