C106B Discussion 9: Grasping

1   Introduction

Grasping objects is a major part of current robotic manipulation research. To approach this problem, we will discuss wrenches and their mathematical properties and then apply them to the idea of contact forces.

2   Wrenches

Last semester, we started our discussion of robotic arm movement talking about *kinematics*. This deals with the different positions and angles our body frame can potentially reach. The orientation of the **B** frame with respect to the **A** frame is given by the forward kinematic map:

$$g_{AB}(\theta_1) = e^{\hat{\xi}_1 \theta_1} g_{AB}(0)$$

We then discussed *kinetics*, which deal with velocities and accelerations. The relative velocity of a point given in the body frame for some angular velocity $\dot{\theta}$ is

$$v_{q_S} = \hat{V}^s_{AB} q_S, \quad V^s_{AB} = \xi' \dot{\theta} = \begin{bmatrix} v^s_{AB} \\ \omega^s_{AB} \end{bmatrix}$$

$$v_{q_B} = \hat{V}^b_{AB} q_B, \quad V^b_{AB} = \xi^\dagger \dot{\theta} = \begin{bmatrix} v^b_{AB} \\ \omega^b_{AB} \end{bmatrix}$$

where ξ' and $\xi^\dagger$ are the current spatial and body twists.

Now, we move to dynamics for an arm - analyzing the relationship between *forces* applied on the body and its motion! A *wrench* follows the same kind of linear/angular form as twists:

$$\Gamma = \begin{bmatrix} f \\ \tau \end{bmatrix}$$

where f is a linear force component, and τ is a torque.

Unlike angular velocities, however, to compute the total torque on some joint, we use the *transpose* of the twist.

$$\tau = \xi'^T \Gamma^S$$

$$\tau = \xi^{\dagger^T} \Gamma^B$$

The difference between the spatial and body wrench Γs is the frame in which we are applying the wrench.

Problem 1: How does a wrench Γ^B applied on the **B** frame affect the torque at joint ξ_1?
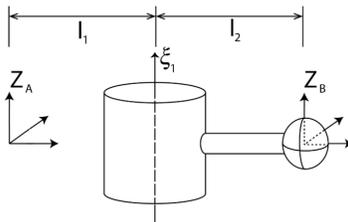
Because we are applying a wrench on frame B, it will look like the following:

$$\Gamma^B = \begin{bmatrix} f_x \\ f_y \\ f_z \\ \tau_x \\ \tau_y \\ \tau_z \end{bmatrix}$$

The torque τ on ξ_1 will be equivalent to $\xi_B^T \Gamma^B$. Recall that a twist ξ is a 6x1 vector composed of $\begin{bmatrix} -\omega \times q \\ \omega \end{bmatrix}$.

Let's compute each part of this vector.

Our ω will be the axis of rotation expressed in B. Since ξ_1 points in the z-direction, we have $\omega = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

The point q is some point on the axis of rotation, also expressed in the B frame. An easy one is $q = \begin{bmatrix} -l_2 \\ 0 \\ 0 \end{bmatrix}$.

If we put the two together, we get $-\omega \times q = \begin{bmatrix} 0 \\ l_2 \\ 0 \end{bmatrix}$.

To calculate our final torque $\tau = \xi^T \Gamma = \begin{bmatrix} 0 & l_2 & 0 & 0 & 0 & 1 \end{bmatrix} \Gamma^B$. You can think of this as an $F \cdot d$

3   Adjoints for Wrenches

Spatial and body velocities are related to one another using adjoints (which are invertible):

$$V_{AB}^S = Ad_{g_{AB}} V_{AB}^b$$

$$Ad_g = \begin{bmatrix} R & \hat{p}R \\ 0 & R \end{bmatrix}$$

Can we figure out a similar relationship for wrenches?

It turns out we can!

$$\Gamma^S = Ad_{g_{AB}^{-1}}^T \Gamma^B$$

Problem 2: The work of a force is calculated by $W = F \cdot d$. Two wrenches are equivalent if they generate the same amount of work. Use this concept to prove the adjoint relationship for wrenches.

Let's express the work equation using velocities instead, as displacement is an integral of velocity. Specifically, $W = \int_{t_1}^{t_2} F \cdot V \, dt = \int_{t_1}^{t_2} \Gamma \cdot V \, dt$.

This holds true for both spatial and body wrenches; as a result, $W = \int_{t_1}^{t_2} \Gamma^S \cdot V \, dt = \int_{t_1}^{t_2} \Gamma^B \cdot V \, dt$

Let's pull out the integrand, which we can do because on an infinitesimal basis, they should be equivalent.

$$\Gamma^S \cdot V^s = \Gamma^{S^T} V^s = \Gamma^B \cdot V^B \tag{1}$$
$$= (\Gamma^B)^T V^B \tag{2}$$
$$= (\Gamma^B)^T Ad_{g_{BS}} V^S \tag{3}$$
$$\tag{4}$$

We can invert V^S on both sides.

$$\Gamma^{S^T} = (\Gamma^B)^T Ad_{g_{BS}} \tag{5}$$
$$\Gamma^S = Ad_{g_{BS}}^T \Gamma^B \tag{6}$$
$$\Gamma^S = Ad_{g_{SB}^{-1}}^T \Gamma^B \tag{7}$$
$$\tag{8}$$

This proves the adjoint relationship between wrenches!

4 Jacobians for Wrenches

Last semester, we used the spatial and body Jacobians to transform individual joint velocities to the end-effector velocity and vice-versa. You also used this concept in Project 1 to generate jointspace trajectory commands. Recall that

$$V^S = J^S(\theta)\dot{\theta}$$
$$V^B = J^B(\theta)\dot{\theta}$$

where $\dot{\theta}$ is a vector of individual joint velocities. The Jacobian itself is a composition of the individual joint twists in their current configuration.

To compute the torques on each joint based on a wrench applied in the spatial or body frame, we can use the Jacobian as well:

$$\begin{bmatrix} \tau_1 \\ \vdots \\ \tau_n \end{bmatrix} = (J^S)^T \Gamma^S = (J^B)^T \Gamma^B$$

Problem 3: Compute the joint torques if we apply a force on the body frame.
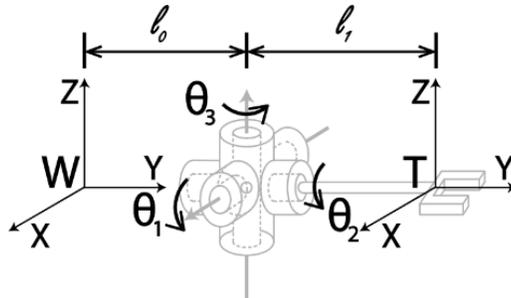
In this figure,

$$\begin{bmatrix} \tau_1 \\ \tau_2 \\ \tau_3 \end{bmatrix} = (J^B)^T \Gamma^B$$

$$w = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, q = \begin{bmatrix} 0 \\ -l_1 \\ 0 \end{bmatrix}$$

$$-w \times q = \begin{bmatrix} 0 \\ 0 \\ l_1 \end{bmatrix}$$

$$J_B^T = \begin{bmatrix} 0 & 0 & l_1 & 1 & 0 & 0 \\ & \cdots & (\xi_2^\dagger)^T & \cdots & \\ & \cdots & (\xi_3^\dagger)^T & \cdots & \end{bmatrix}$$

5   Grasp map

When going into robotic hands, we want to actually grab objects. One *contact* is defined by

$$F_{c_i} = B_{c_i} f_{c_i}$$

Where B is the contact basis, or the directions in which the contact can apply force, and f is a vector in that basis (the actual forces being applied). F is the 6x1 wrench which the contact applies. In our case, we use a soft contact model, which has both lateral and torsional friction components, so the basis is

$$B_{c_i} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

However, in the real world, friction is not infinite. For the contact to resist a wrench without slipping, the contact vector must lie within the *friction cone*, which is defined

$$FC_{c_i} = \{f \in \mathbb{R}^4 : \sqrt{f_1^2 + f_2^2} \leq \mu f_3, f_3 > 0, |f_4| \leq \gamma f_3\}$$

f_3 is the amount of normal force being applied, f_1 and f_2 are the forces in the other two perpendicular directions, and f_4 is a torque. The friction cone therefore tells us the forces that can be applied onto an object that would be resisted by this contact.

However, we want the wrenches that a contact point can resist in the world frame, not the contact frame. So we use the adjoint to transform the contact basis:

$$G_i := \begin{bmatrix} R_{oc_i} & 0 \\ \widehat{p}_{oc_i} R_{oc_i} & R_{oc_i} \end{bmatrix} B_{c_i} = Ad_{g_{co_i}^{-1}}^T B_{c_i}$$

A grasp is a *set* of contacts (maybe multiple fingers in a hand or the two sides of a Sawyer gripper), so we define the wrenches (in the world frame) a grasp can resist as:

$$F_o = G_1 f_{c_1} + \cdots + G_k f_{c_k} = \begin{bmatrix} G_1 & \cdots & G_k \end{bmatrix} \begin{bmatrix} f_{c_1} \\ \vdots \\ f_{c_k} \end{bmatrix} = Gf$$

The resulting compound matrix G above is called the *grasp map*, summing up multiple forces.

6   Force closure

A grasp is in *force closure* when finger forces lying in the friction cones span the space of object wrenches

$$G(FC) = \mathbb{R}^6$$

Essentially, this means that any external wrench applied to the object can be countered by the sum of contact forces (provided the contact forces are high enough).

For a two-contact soft-fingered grasp, we also have the following theorem which makes it very easy to check when a grasp is in force closure. This is theorem 5.7 from MLS.

Theorem. *A spatial grasp with two soft-finger contacts is force-closure if and only if the line connecting the contact point lies inside both friction cones.*
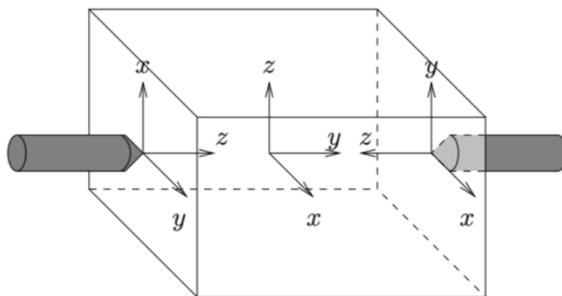
Figure 1: Two finger grasp.

6.1   Discretizing the Friction Cone

Checking that $f \in FC$ can be difficult. Often when evaluating grasps, we will write down an optimization problem that has $f \in FC$ as a constraint.

$$FC_{c_i} = \begin{cases} \sqrt{f_1^2 + f_2^2} \leq \mu f_3 \\ f_3 > 0 \\ |f_4| \leq \gamma f_3 \end{cases}$$

We can approximate the (conical) friction cone as a pyramid with n vertices. The level sets of the friction cone are circles, but the level sets for this convex approximation are n sided polygons circumscribed by the circle. Thus, the interior of this convexified friction cone is a conservative approximation of the friction cone itself.
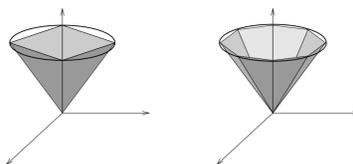
Figure 2: Approximations of the friction cone. From section 5.3 of MLS.

Any point in the interior of this pyramid can be described as a sum of

$$f = \alpha_0 f_0 + \sum_{i=1}^{n} \alpha_i f_i = F\alpha$$

where f_i are the edges of the pyramid and f_0 a straight line in z, and the weights α are all non-negative. Here, we can write a composite matrix F (different from the F above!) with the f_i vectors as its columns. This lets us more easily characterize any f in the friction cone. We make the approximation that $f \in FC$ if and only if there exists a non-negative vector α such that $f = F\alpha$.

With this approximation, the condition that $f \in FC$ is equivalent to the pair of linear constraints $\{f = F\alpha, \alpha \geq 0\}$ (where this inequality is understood to be element-wise).

Problem 4:
Let w be a given wrench. Let a two-contact grasp be given to you with contact grasp maps G_1 and G_2. We wish to find the input force $f \in FC$ with the smallest norm that can resist the wrench w applies at the center of mass of the object being grasped. Using the polyhedral approximation of the friction cone, write this as a quadratic program.

This is the quintessential grasping optimization problem. How do we find out how to optimize the forces with which we are holding our object? In this particular case, we are given our grasp maps, so we know the contact points. Usually, those are sampled, and the best set of contact points with the best set of forces is used for our final output. This is just one way to find grasps - you can perform research on the Ferrari-Canny metric or Robust Force Closure for other approaches.

We want to minimize the force that we need to apply to hold our object. As a result, the optimization objective is **min $f^T f$**.

Now, we determine our constraints for our QP solver. Our f vector is a composition of 2 inner vectors, one for each contact point: $f = \begin{bmatrix} f_a \\ f_b \end{bmatrix}$. Each f is a composition of the x, y, and z elements.

We want our normal forces to be positive. As a result, $f_{a,z}, f_{b,z} \geq 0$.

We want to ensure that our forces lie within our friction cone when we're optimizing. We can therefore write out the constraints $f_a = F\alpha_a$ and $f_b = F\alpha_b$ with $\alpha_a, \alpha_b \geq 0$.

Finally, we want to best oppose the wrench in the problem and want to apply a force that can do so. Thus, we get $-w = Gf$, where Gf applies the grasp map on our contact forces and determines how well it is holding the object. We negate w because we want to provide an opposing force.

Problem 5:
Consider the box grasped by 2 soft-finger contacts shown in the figure above. Find the grasp map. Assume the object is a cube of side-length 2.

Let's find the rotation and position matrix for each of the contacts, and we can aplpy our grasp map formula directly.
Contact 1:

$$R_{c_1} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, p_{c_1} = \begin{bmatrix} 0 \\ -r \\ 0 \end{bmatrix}$$

Contact 2:

$$R_{c_2} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}, p_{c_2} = \begin{bmatrix} 0 \\ r \\ 0 \end{bmatrix}$$

Each G_i can be found in the following way:

$$G_i = \begin{bmatrix} R_{oc_i} & 0 \\ \hat{p}_{oc_i} R_{oc_i} & R_{oc_i} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The final G matrix we end up with is

$$
G = \begin{bmatrix}
0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 \\
1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & r & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 \\
0 & r & 0 & 0 & -r & 0 & 0 & 0
\end{bmatrix}
$$

This grasp map would be multiplied by the force vector of each contact to determine the wrenches that can be resisted.