C106B Discussion 5: Kinematic Constraints

1 Introduction

Today, we'll talk about:

1. Pfaffian Constraints
2. Equivalent Control Systems
3. Lie Brackets & Controllability

2 Pfaffian Constraints

When performing path planning tasks in robotics, it's essential to have an understanding of how our system moves, as we *always* want to generate paths that are feasible for our system to follow! It's therefore important for us to understand the *kinematic constraints* on a system - the constraints that impact the possible positions and velocities of the system.
Let's consider a physical system with generalized coordinates $q_1, q_2, ..., q_n$. We know that using Lagrangian mechanics, we can find the dynamics of the system by computing n differential equations:

$$\frac{d}{dt}\Big(\frac{\partial L}{\partial \dot{q}_i}\Big) - \frac{\partial L}{\partial q_i} = F_i \tag{1}$$

A *kinematic constraint* imposes restrictions on the generalized coordinates and their velocities. A *Pfaffian constraint* is a constraint of the form:

$$\omega_i(q)\dot{q} = 0 \tag{2}$$

Where q is a vector of the system's generalized coordinates. A Pfaffian constraint on the velocities of q_i is said to be *integrable* if it is equivalent to a constraint on the *positions* of q_i:

$$\omega_i(q)\dot{q} = 0 \Longleftrightarrow h_i(q) = 0 \tag{3}$$

If a set of k Pfaffian constraints ω_i are all integrable, the set of constraints is said to be *holonomic*. If a subset of the constraints are integrable, then the constraints are said to be *partially nonholonomic*. If no constraints are integrable, the set is *completely nonholonomic*.

Problem 1: A uniform, rigid pendulum of length $2L$ swings about a pivot point. The angle of the pendulum to the vertical is θ and the position of the center of mass is (x, y). Write the constraints on the values of x, y subject to the pendulum's motion. Are these constraints holonomic?

Solution: Assuming an x coordinate which points vertically downward from the pendulum pivot point, and a y coordinate which points horizontally to the right of the pendulum pivot, we may express the constraints on the pendulum's motion as:

$$x - L\cos\theta = 0 \tag{4}$$
$$y - L\sin\theta = 0 \tag{5}$$

Since these constraints are purely in terms of the generalized coordinates x, y, θ, and don't involve their derivatives, these constraints must be holonomic.

3 Equivalent Control Systems

Suppose we have k independent, nonholonomic Pfaffian constraints $\omega_1(q)\dot{q} = 0, ..., \omega_k(q)\dot{q}$, where $q \in \mathbb{R}^n$ is a vector of generalized coordinates. We can write these constraints in matrix form $A(q)\dot{q} = 0$ as:

$$\begin{bmatrix} - & \omega_1(q) & - \\ & \vdots & \\ - & \omega_k(q) & - \end{bmatrix} \dot{q} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix} \tag{6}$$

We know that the columns of $A(q)$ above are linearly independent, and that all allowable trajectories $\dot{q}$ must be in the null space of $A(q)$. Since $q \in \mathbb{R}^n$, and the matrix is in $\mathbb{R}^{k \times n}$ with k independent constaints, the null space must be $n - k = m$ dimensional.
Therefore, there exist $g_1(q), ..., g_m(q)$ that span the basis of this null space such that:

$$\dot{q} = u_1 g_1(q) + ... + u_m g_m(q) \tag{7}$$

Are all allowable trajectories, where $u_i \in \mathbb{R}$ are scalars. Since we can arbitrarily control u_i, we have found an *equivalent control system* for our dynamics just using the kinematic constraints. This equivalent control system is a simpler model that expresses what it means for a trajectory to be allowable, and we can use it to control our system's generalized coordinates. Note that each $g_i(q) \in \mathbb{R}^n$ is called a *vector field*, as it maps a vector to a vector.

Problem 2: The Raibert hopper, which has generalized coordinates $q = [\phi, l, \theta]^T$ has the following nonholonomic constraint on its dynamics.

$$I\dot{\theta} + m(l+d)^2(\dot{\theta} + \dot{\phi}) = 0 \tag{8}$$

Rewrite this constraint in the form $A(q)\dot{q} = 0$, and find a basis for the null space of $A(q)$.

Solution: Exanding this expression, we see:

$$I\dot{\theta} + m(l+d)^2\dot{\theta} + m(l+d)^2\dot{\phi} = 0 \tag{9}$$

Let's factor out a $\dot{q} = [\dot{\phi}, \dot{l}, \dot{\theta}]$ from this expression. This leaves us with:

$$\begin{bmatrix} m(l+d)^2 & 0 & m(l+d)^2 + I \end{bmatrix} \begin{bmatrix} \dot{\phi} \\ \dot{l} \\ \dot{\theta} \end{bmatrix} = 0 \tag{10}$$

This is an expression of the form $A(q)\dot{q} = 0$! Now, let's find a basis for the null space of $A(q)$. We may pick:

$$g_1 = \begin{bmatrix} \frac{1}{m(l+d)^2} \\ 0 \\ \frac{-1}{m(l+d)^2+I} \end{bmatrix}, \; g_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \tag{11}$$

As our basis.

4 Lie Brackets & Controllability

How can we use a nonholonomic constraint $\dot{q} = A(q)q, q \in \mathbb{R}^n$ to design feedback controllers and path planners for our system? Let's imagine that we want to drive our system to the position $q_d \in \mathbb{R}^n$? Under our kinematic constraints, is it actually possible to steer our system to q_d? To answer this question, we'll need a few tools from the field of differential geometry.

Definition 1 ***Lie Bracket***
The Lie bracket of two vector fields $f(q), g(q)$ is defined:

$$[f, g](q) = \frac{\partial g}{\partial q} f(q) - \frac{\partial f}{\partial q} g(q) \tag{12}$$

The Lie bracket measures whether flows of equal time along f and g commute.

Definition 2 ***Lie Algebra***
The Lie algebra of a set of vector fields $\{g_1, g_2\}$, denoted $\mathcal{L}(g_1, g_2)$, is the span of all linear combinations of g_1, g_2, their Lie brackets, and higher order Lie brackets:

$$g_1, g_2, [g_1, g_2], [g_1, [g_1, g_2]], [g_2, [g_1, g_2]], \ldots \tag{13}$$

For a set of m vector fields, $g_1, ..., g_m$, the Lie algebra $\mathcal{L}(g_1, ..., g_m)$ is similarly defined by taking the span of the vector fields and their Lie brackets with each other.

Here's the basic concept of our big idea for this section: if the vector fields $g_1(q), ..., g_m(q)$ from our equivalent control system have *nonzero* Lie brackets, we might be able to form a basis of vector fields we may travel along to reach any location.

Theorem 1 ***Small Time Local Controllability***
A system is small time locally controllable at a point q_0 if the set of states the system can reach in finite time starting from q_0 forms a ball around q_0. If the dimension of $\mathcal{L}(g_1, ..., g_m)$ is equal to the dimension of q, and the positive span of the vector $[u_1, ..., u_m]$ is $\mathbb{R}^m$, then the system:

$$\dot{q} = g_1(q)u_1 + ... + g_m(q)u_m \tag{14}$$

Is small time locally controllable.

Problem 3: Imagine we have a vector of generalized coordinates $q = [x, y, z]^T$. These coordinates have a kinematic constraint which may be represented by the control system:

$$\dot{q} = \begin{bmatrix} 1 \\ 0 \\ y \end{bmatrix} u_1 + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} u_2 = g_1(q)u_1 + g_2(q)u_2 \tag{15}$$

Where u_1 and u_2 can have any values in $\mathbb{R}$. Find the Lie bracket $[g_1, g_2]$, conclude the Lie algebra $\mathcal{L}(g_1, g_2)$ has dimension 3, and show that the system is small time locally controllable.

Solution: First, we find the Lie bracket, $[g_1, g_2]$. From the definition of the Lie bracket:

$$[g_1, g_2] = \frac{\partial g_2}{\partial q} g_1 - \frac{\partial g_1}{\partial q} \partial g_2 \tag{16}$$

Let's calculate the different partial derivatives in this expression.

$$\frac{\partial g_2}{\partial q} = \begin{bmatrix} \frac{\partial g_{21}}{\partial x} & \frac{\partial g_{21}}{\partial y} & \frac{\partial g_{21}}{\partial z} \\ \frac{\partial g_{22}}{\partial x} & \frac{\partial g_{22}}{\partial y} & \frac{\partial g_{22}}{\partial z} \\ \frac{\partial g_{23}}{\partial x} & \frac{\partial g_{23}}{\partial y} & \frac{\partial g_{23}}{\partial z} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{17}$$

$$\frac{\partial g_1}{\partial q} = \begin{bmatrix} \frac{\partial g_{11}}{\partial x} & \frac{\partial g_{11}}{\partial y} & \frac{\partial g_{11}}{\partial z} \\ \frac{\partial g_{12}}{\partial x} & \frac{\partial g_{12}}{\partial y} & \frac{\partial g_{12}}{\partial z} \\ \frac{\partial g_{13}}{\partial x} & \frac{\partial g_{13}}{\partial y} & \frac{\partial g_{13}}{\partial z} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \tag{18}$$

$$\tag{19}$$

Now, multiplying out, we get the Lie bracket:

$$[g_1, g_2] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ y \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix} \tag{20}$$

Now, this gives us the set of vector fields:

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ y \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix} \right\} = \{g_1, g_2, [g_1, g_2]\} \tag{21}$$

This set is linearly independent! Since the Lie algebra, $\mathcal{L}(g_1, g_2)$ is the span of g_1, g_2, and their Lie brackets, we conclude that the Lie algebra is three dimensional (has rank 3). Since we may choose arbitrary inputs to the system and the Lie algebra is of rank 3 for a three-dimensional state vector, we conclude the system is small time locally controllable.