C106B Discussion 3: Feedback Linearization

1 Introduction

Today, we'll talk about:

1. SISO Feedback Linearization
2. Relative Degree
3. MIMO Feedback Linearization

2 SISO Feedback Linearization

Linear systems, of the form $\dot{x} = Ax + Bu, y = Cx$, are the simplest form of dynamical system to analyze and control. Can we use an input, u, to make a control affine nonlinear system:

$$\dot{x} = f(x) + g(x)u \tag{1}$$
$$y = h(x) \tag{2}$$

Where $g(x) \neq 0$, behave like a linear system? Let's consider the single input, single output (SISO) case, where $u, y \in \mathbb{R}, x \in \mathbb{R}^n$. Let's try focusing on the output of the system, y, the variable we actually wish to control. Computing its derivative along the trajectories of the system:

$$\dot{y} = \frac{\partial h}{\partial x} f(x) + \frac{\partial h}{\partial x} g(x)u \tag{3}$$

The terms $\frac{\partial h}{\partial x} f(x)$ and $\frac{\partial h}{\partial x} g(x)$ are called **Lie derivatives** (pronounced "Lee" derivatives).

$$L_f h(x) = \begin{bmatrix} \frac{\partial h}{\partial x_1} & \dots & \frac{\partial h}{\partial x_n} \end{bmatrix} \begin{bmatrix} f_1(x) \\ \vdots \\ f_n(x) \end{bmatrix} \tag{4}$$

The Lie derivative of $h(x)$ along $f(x)$ tells us how quickly h changes along the vector field $f(x)$ - we can think of it like an inner product between $\frac{\partial h}{\partial x}$ and $f(x)$. Using Lie derivative notation:

$$\dot{y} = L_f h(x) + L_g h(x)u \tag{5}$$

Problem 1: Consider the SISO system above. Assuming $L_g h(x) \neq 0$ in our region of interest, find a formula for u such that when u is plugged into the output dynamics, the following equation results:

$$\dot{y} = v \tag{6}$$

Where $v \in \mathbb{R}$ is an arbitrary scalar we can control.
Solution: Assuming the input term is nonzero, we can choose:

$$u = \frac{1}{L_g h}(-L_f h + v) \tag{7}$$

Plugging this into the output dynamics, we observe that we get the relationship $\dot{y} = v$.

3 Relative Degree

What if $L_g h(x) = 0$ in the SISO case? We can try taking *higher* derivatives of y until $L_g h(x) \neq 0$ and our input term appears. The smallest level of derivative for which the input will appear in the derivative of y is called the **relative degree**, r, of the system.[1]
Assuming that the input does not show up for all derivatives less than r, how can we express the r^{th} derivative of the output using Lie derivative notation?

$$y^{(r)} = L_f^r h(x) + L_g L_f^{r-1} h(x) u \tag{8}$$

By the definition of relative degree, at the r^{th} derivative, $L_g L_f^{r-1} h(x) \neq 0$ in our region of interest. This means that we can pick a feedback linearizing control law:

$$u = \frac{1}{L_g L_f^{r-1} h(x)}(-L_f^r h(x) + v) \Rightarrow y^{(r)} = v \tag{9}$$

Where v is an arbitrary scalar. Note that as a general rule of thumb for SISO systems, $r \leq n$ if $g(x) \neq 0$.

Problem 2: Consider a simple pendulum which swings under gravity with some friction. A torque, which we can control, is applied to the pendulum at its pivot point. The equations of motion of this system may be written as a control affine, SISO system in state space as:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ -\frac{g}{l}\sin x_1 - \beta x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \tag{10}$$

$$y = x_1 \tag{11}$$

Where $[x_1, x_2] = [\theta, \dot{\theta}]$ and $u = \tau$, the torque applied to the pendulum. The output of this system is $y = x_1$, the angle of the pendulum. Take the time derivatives of the output along the trajectories of the system until the input, u, appears. What is the relative degree of the system? *Hint: Here, it's easier to take the time derivative directly instead of using Lie derivative notation.*

Problem 3: Find a feedback linearizing input u to the pendulum system such that when the input is applied to the system, the following differential equation:

$$\ddot{y} = v \tag{12}$$

Where $v \in \mathbb{R}$ is an arbitrary scalar, governs the dynamics.
Solution: We begin by taking the first time derivative of the output:

$$\dot{y} = \dot{x}_1 = x_2 \tag{13}$$

No input term has appeared yet, so we take another derivative:

$$\ddot{y} = \dot{x}_w = -\frac{g}{l}\sin x_1 - \beta x_2 + u \tag{14}$$

Now, we have an input appearing! To get the desired form, we choose:

$$u = \frac{g}{l}\sin x_1 + \beta x_2 + v \tag{15}$$

[1] Note that relative degree is actually defined at a particular point in the domain.

4 MIMO Linearization

How can we generalize our feedback linearization results to a multi input, multi output (MIMO) system? We will consider *square* MIMO systems, which have the same number of inputs as outputs:

$$\dot{x} = f(x) + g(x)u, \; x \in \mathbb{R}^n, u \in \mathbb{R}^m \tag{16}$$
$$y = h(x), \; y \in \mathbb{R}^m \tag{17}$$

MIMO feedback linearization will *largely* be the same as SISO feedback linearization! We can think about the output y, as a collection of single outputs:

$$y = \begin{bmatrix} y_1 \\ \vdots \\ y_m \end{bmatrix} = \begin{bmatrix} h_1(x) \\ \vdots \\ h_m(x) \end{bmatrix} \tag{18}$$

How can we express the time derivatives of these outputs using Lie derivative notation? In this case, instead of being a vector, $g(x)$ will be a matrix:

$$g(x) = \begin{bmatrix} | & & | \\ g_1(x) & \dots & g_m(x) \\ | & & | \end{bmatrix} \tag{19}$$

Problem 4: Show that the first time derivative of the output y_j along the trajectories of the system is computed:

$$\dot{y}_j = L_f h_j(x) + \sum_{i=1}^{m} L_{g_i} h_j(x) u_i \tag{20}$$

Hint: Begin by applying the chain rule, try multiplying $\frac{\partial h_j}{\partial x}$ by each column of $g(x)$ to get the sum!

$$\dot{y}_j = \frac{\partial h_j}{\partial x}(\dot{x}) \tag{21}$$
$$= \frac{\partial h_j}{\partial x}(f(x) + g(x)u) \tag{22}$$
$$= \frac{\partial h_j}{\partial x} f(x) + \frac{\partial h_j}{\partial x} g(x)u \tag{23}$$
$$= L_f h_j(x) + \frac{\partial h_j}{\partial x} \begin{bmatrix} | & & | \\ g_1(x) & \dots & g_m(x) \\ | & & | \end{bmatrix} \begin{bmatrix} u_1 \\ \vdots \\ u_m \end{bmatrix} \tag{24}$$
$$= L_f h_j(x) + \begin{bmatrix} | & & | \\ \frac{\partial h_j}{\partial x} g_1(x) & \dots & \frac{\partial h_j}{\partial x} g_m(x) \\ | & & | \end{bmatrix} \begin{bmatrix} u_1 \\ \vdots \\ u_m \end{bmatrix} \tag{25}$$
$$= L_f h_j(x) + \begin{bmatrix} | & & | \\ L_{g_1} h_j & \dots & L_{g_m} h_j \\ | & & | \end{bmatrix} \begin{bmatrix} u_1 \\ \vdots \\ u_m \end{bmatrix} \tag{26}$$
$$= L_f h_j(x) + \sum_{i=1}^{m} L_{g_i} h_j(x) u_i \tag{27}$$